From Discrete Time to Continuous Time Modeling

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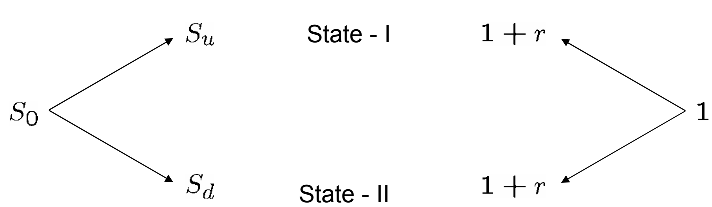
Arrow-Debreu Securities

- Consider a simple one-period economy containing a single stock and a single savings account.
- The economy has two possible states in one time step from now.
- Is the model arbitrage free?



Stock Evolution

Bank Account Evolution

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- An arbitrage portfolio is a portfolio which costs zero at t=0, but may have a positive pay-off at t=1
- The model is arbitrage free if and only if

$$S_d < (1+r)\,S_0 < S_u$$

  - Suppose that $\mathbf{S_u > S_d \geq (1+r)\,S_0}$, then it is always better to invest in the asset than the money-market account
  - An example of an arbitrage portfolio is
    - Long **1** unit of Asset
    - Short $\mathbf{1/S_0}$ units of the Bank Account

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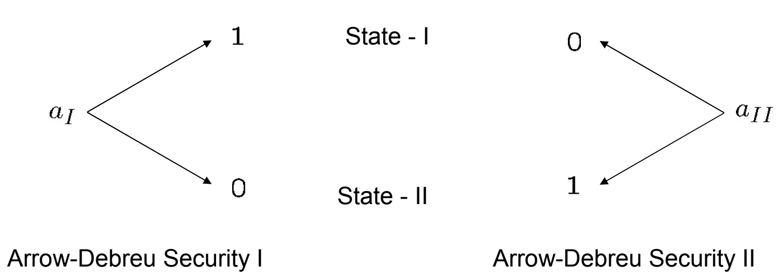
- Consider two fictitious assets which pay exactly 1 in one of the two states of the world and zero in the other.
- What is a rational price for these assets?



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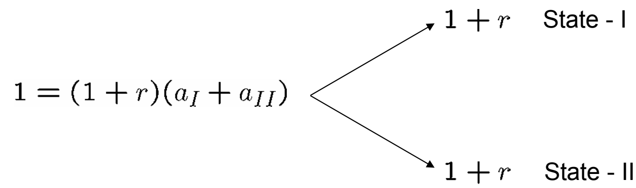
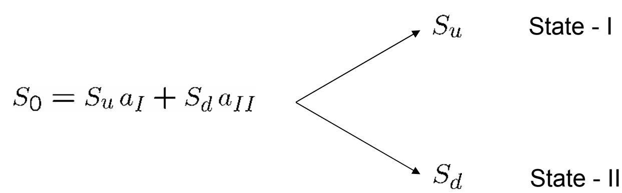
- Make a portfolio of AD securities which generate the pay-offs of the existing claims:





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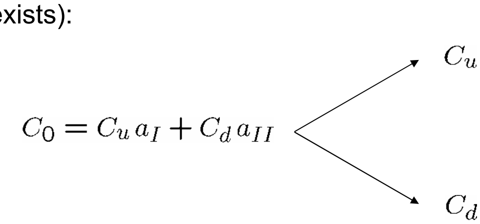
- Solving the linear system gives the prices of the AD securities:

$$a_I = \frac{1}{1+r} \frac{(1+r)\, S_0 - S_d}{S_u - S_d}$$

$$a_{II} = \frac{1}{1+r} \frac{S_u - (1+r)\, S_0}{S_u - S_d}$$

- Given these prices, the price of a contingent claim paying $\mathbf{C_u}$ and $\mathbf{C_d}$ in the two states of the world must be (otherwise an arbitrage exists):



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- The price of the claim can be rewritten as follows:

$$C_0 = C_u a_I + C_d a_{II}$$
$$= \frac{1}{1+r} \{q\, C_u + (1-q)\, C_d\}$$

where,

$$q = \frac{(1+r)S_0 - S_d}{S_u - S_d}$$

- Notice that the state probabilities do not appear in price!
- Is **q** a probability?
- YES - since no arbitrage requires $S_d < (1+r)\, S_0 < S_u$

Arrow-Debreu Securities

- Introduce the relative price of an asset/claim as, $\tilde{C}_t \equiv \frac{C_t}{M_t}$
- Where $\mathbf{M_t}$ denotes the value of the money-market account (bank account) at time **t** and is equal to $\mathbf{(1+r)^t}$
- Then,

$$\tilde{C}_0 = \mathbf{E}^{Q}[\tilde{C}_1] = q\tilde{C}_u + (1-q)\tilde{C}_d$$

- This is implies that the relative prices of assets have zero expected change under the probability measure $\mathcal{Q}$.
- Random variables of this type are called **Martingales**

Hedge-based Pricing

- A dual, yet equivalent, method for determining prices is by utilizing a hedging strategy

- Consider a portfolio set up at **t = 0** consisting of:
  - **x** units of stock
  - **y** units of the bank account

- At time t = 1 this portfolio will be worth
  - $\mathbf{x\, S_u + y\,(1+r)}$ in state-I
  - $\mathbf{x\, S_d + y\,(1+r)}$ in state-II

- It is possible to choose **x** and **y** such that the pay-off of the contingent is matched exactly regardless of which state prevails at time **t=1**

Hedge-based Pricing

- This leads to the linear system

$$\begin{aligned} C_u &= x\,S_u + y\,(1+r) \\ C_d &= x\,S_d + y\,(1+r) \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= \frac{C_u - C_d}{S_u - S_d} \\ y &= \frac{1}{1+r}\,\frac{C_d\,S_u - C_u\,S_d}{S_u - S_d} \end{aligned}$$

- Since the pay-off of the claim and the portfolio are identical, they must have the same price today,

$$C_0 = xS_0 + y = \frac{1}{1+r}\,\{q\,C_u + (1-q)\,C_d\}$$

Multi-Period Binomial Model

- This model extends naturally to multiple periods
- Let $x_0, x_1, x_2, \cdots$ represent a set of identical independent random Bernoulli variables with,

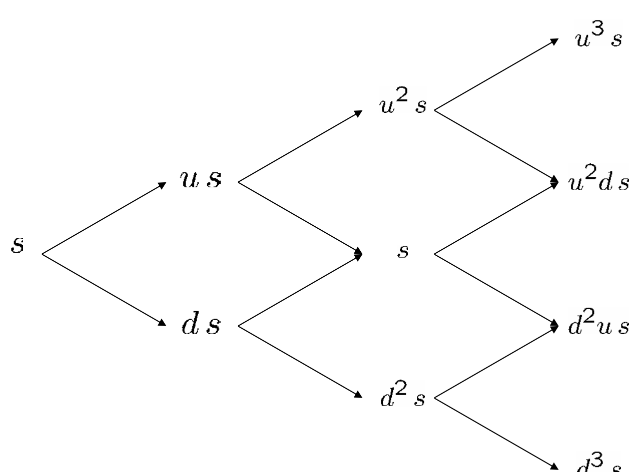
$$\mathbb{P}(x_t = +1) = p \quad \text{and} \quad \mathbb{P}(x_t = -1) = 1 - p \qquad \forall t = 0, 1, \ldots$$

- Then assume that the asset price dynamics satisfies,

$$S_t = S_{t-1} \; \exp\{\alpha \, x_{t-1}\}$$

- That is, the asset has a (continuously compounded) return of $\pm\,\alpha$ each period.
- Such dynamics can be represented by a recombining tree as shown on the next slide

Multi-Period Binomial Model



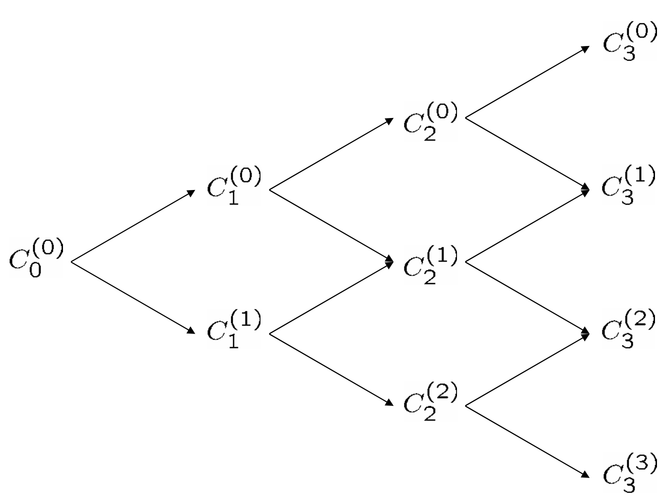
with $\quad u = \dfrac{1}{d} = \exp\{\alpha\}$

Multi-Period Binomial Model

- Prices of European contingent claims can be obtained through backwards recursion
- The claim will define the pay-off at maturity
- Use the discounted expectation in the risk-neutral measure (the probability **p** is replaced by **q**) to compute the prices on the nodes one-time prior
- Repeat the process until time **t = 0** is reached.
- Denoting the price of the claim at time t with asset level $\mathbf{S_t = S_0\, u^{t-2j}}$ by $\mathbf{C_t^{(j)}}$ the recursion formula can be compactly written as follows:

$$C_t^{(j)} = \frac{1}{1+r}\left(q\, C_{t+1}^{(j)} + (1-q)\, C_{t+1}^{(j+1)}\right)$$

Multi-Period Binomial Model



Multi-Period Binomial Model

- The hedging strategy at each node can also be obtained in a similar manner

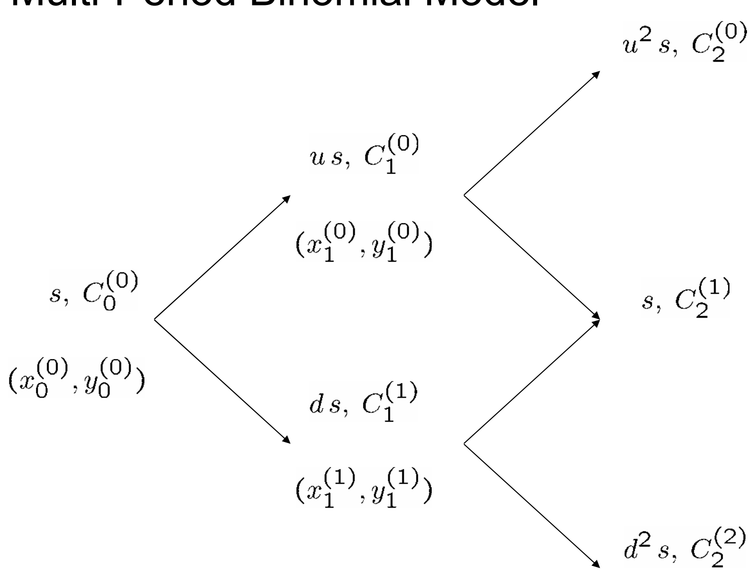
$$x_t^{(j)} = \frac{1}{S_0} \frac{C_{t+1}^{(j)} - C_{t+1}^{(j+1)}}{u^{t-2j+1} - u^{t-2j-1}}$$

$$y_t^{(j)} = \frac{1}{1+r} \frac{C_{t+1}^{(j+1)}\, u^{t-2j+1} - C_{t+1}^{(j)}\, u^{t-2j-1}}{u^{t-2j+1} - u^{t-2j-1}}$$

- Of course the relationship between the hedging parameters and the price still holds

$$C_t^{(j)} = x_t^{(j)}\, S_0\, u^{t-2j} + y_t^{(j)}$$

Multi-Period Binomial Model



Multi-Period Binomial Model

- It is possible to price the claim without resorting to computing the value at every single node..
- Recall that the asset price dynamics is given by

$$S_t = S_{t-1}\ \exp\{\alpha\, x_{t-1}\}$$

- Where (in the risk neutral measure)

$$\mathbb{P}(x_t = +1) = q \quad \text{and} \quad \mathbb{P}(x_t = -1) = 1 - q \qquad \forall t = 0, 1, \ldots$$

- For a European claim the pay-off function depends only on the terminal value of the asset, $\mathbf{S_T}$, but,

$$\begin{aligned} S_T &= S_0\ \exp\{\alpha\,(x_0 + x_1 + \ldots + x_{T-1})\} \\ &= S_0\ \exp\{\alpha\, X_T\} \end{aligned}$$

- Where $\mathbf{X_T = x_0 + x_1 + \ldots + x_{T-1}}$ and is binomial random variable of degree T and success probability q.

Multi-Period Binomial Model

- Consequently, the price of a European contingent claim, with payoff function $\boldsymbol{\phi(S_T)}$, in the binomial model is,

$$\begin{aligned} C_0 &= \frac{1}{(1+r)^T}\,\mathbf{E}^{\mathbb{Q}}\left[\phi(S_T)|\mathcal{F}_0\right] \\ &= \frac{1}{(1+r)^T}\,\mathbf{E}^{\mathbb{Q}}\left[\phi(S_0\, e^{\alpha X_T})|\mathcal{F}_0\right] \\ &= \frac{1}{(1+r)^T}\sum_{n=0}^{T}\binom{T}{n} q^n\,(1-q)^{T-n}\,\phi(S_0\, e^{n\alpha}) \end{aligned}$$

- This is the Cox-Ross-Rubinstein (CRR) representation for the price of a European option

Volatility matching

- The parameter α can be specified through the volatility of the asset dynamics.
- In particular, the asset will be forced to have a variance of σ^2 **Δt** (where **Δt** is size of the time step in the tree) when **Δt << 1**
- This leads to system of two equations (one for risk-neutrality and one for variance matching)

$$\mathbf{E}^{Q}\left[S_{t+\Delta t}\middle|\mathcal{F}_t\right] = (1+r\Delta t)\,S_t$$

$$\text{Var}^{Q}\left[\ln\left(\frac{S_{t+\Delta t}}{S_t}\right)\middle|\mathcal{F}_t\right] = \sigma^2\Delta t$$

- The solution when **Δt << 1** can be expressed as,

$$q = \frac{1}{2}\left(1+\frac{r-\frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right) \qquad \text{and} \qquad \alpha = \sigma\sqrt{\Delta t}$$

Continuous Time Limit

- Recall that the asset dynamics was,

$$S_t = S_{t-1}\ \exp\{\alpha\, x_{t-1}\}$$
$$\Rightarrow \quad S_t = S_0\ \exp\{\alpha\,(x_0+x_1+\ldots+x_{t-1})\}$$

- Letting $\mathbf{X_t = x_0 + x_1 + \ldots + x_{t-1}}$. Then, since $\mathbf{x_0, x_1, .., x_{t-1}}$ are all i.i.d. Bernoulli random variables,

$$\mathbf{E}^{Q}[X_t|\mathcal{F}_0] = n\alpha\,\mathbf{E}^{Q}[x_0] = \left(r-\frac{1}{2}\sigma^2\right)\,n\,\Delta t$$
$$\text{Var}^{Q}[X_t|\mathcal{F}_0] = n\alpha^2\,\text{Var}^{Q}[x_0] = \sigma^2\,n\,\Delta t$$

- Furthermore, the central limit theorem says that $\mathbf{X_t}$ is a normal r.v.
- Finally, $\mathbf{X_t - X_{t+s}}$ (for **s > 0**) has a distribution that is independent of **t** (it depends only on **s**.)

Continuous Time Limit

- We can therefore summarize the properties of the asset dynamics as follows:

$$S_t = \exp\{X_t\}$$

- Where, X_t has the following characteristics:
  - Starts at 0
    - $X_0 = 0$
  - Has independent increments
    - $X_t - X_s$ is independent of $X_v - X_u$ whenever $(t,s) \cap (u,v) = \varnothing$
  - Has stationary increments
    - $X_t - X_{t+s} \sim N((r - ½ \sigma^2)\, s;\ \sigma^2\, s)$

- The above properties describe a stochastic process known as **Brownian motion** (or a Weiner process.)

Continuous Time Limit

- The price of a European contingent claim using the Brownian motion representation of the asset dynamics in the continuous time is,

$$\begin{aligned} C_t &= e^{-r(T-t)}\,\mathbf{E}^{\mathbb{Q}}\left[\phi(S_T)|\mathcal{F}_t\right] \\ &= e^{-r(T-t)}\,\mathbf{E}^{\mathbb{Q}}\left[\phi(S_t\, e^{X_T - X_t})|\mathcal{F}_t\right] \\ &= e^{-r(T-t)}\int_{-\infty}^{\infty}\phi(S_t\, e^{x})\, f(x; r, \sigma, T-t)\, dx \end{aligned}$$

- Where **f(x; r,σ, T-t)** represents the normal distribution with appropriate mean and variance,

$$f(x; r, \sigma, T-t) = \frac{\exp\left\{-\frac{1}{2}\frac{\left(x-\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2}{\sigma^2(T-t)}\right\}}{\sqrt{2\pi\sigma^2(T-t)}}$$

The Black-Scholes Pricing Formula

- When the pay-off function is that of a call option, **max(S_T - K, 0)**, the integral can be carried out explicitly,

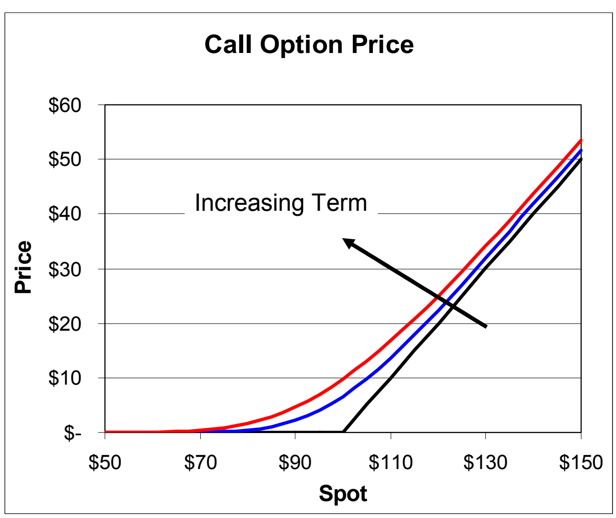
$$\begin{aligned} C_t(S_t; K, T) &= e^{-r(T-t)} \int_{-\infty}^{\infty} \max(S_t\, e^x - K, 0)\, f(x; r, \sigma, T-t)\, dx \\ &= e^{-r(T-t)} \int_{\ln \frac{K}{S_t}}^{\infty} (S_t\, e^x - K)\, f(x; r, \sigma, T-t)\, dx \\ &= S_t\, \Phi(d_+) - e^{-r(T-t)} K\, \Phi(d_-) \end{aligned}$$

- Where,

$$d_\pm = \frac{\ln \frac{S_t}{K} + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

- and **Φ(x)** is the cumulative density function of a standard normal r.v.

The Black-Scholes Pricing Formula



The Black-Scholes Pricing Formula

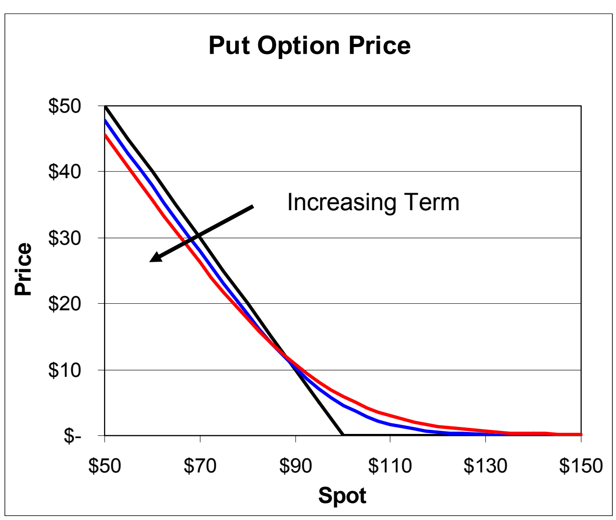
- The price of a put option can be obtained in a similar manner, or through put-call parity,

$$C_t(S_t; K, T) - P_t(S_t; K, T) \;=\; S_t - e^{-r(T-t)}\, K$$

- The result is,

$$P_t(S_t; K, T) = e^{-r(T-t)} K \; \Phi(-d_-) - S_t \; \Phi(-d_+)$$

The Black-Scholes Pricing Formula



Stochastic Integrals

- We would like to define integrals w.r.t. the stochastic variable **W(t)** where **W(t)** is a standard Brownian process
- Consider a simple stochastic process (piecewise constant) **g(s)** with jump points at $a < t_0 < t_1 < \ldots < t_n < b$
- The integral of such a process w.r.t to **W(t)** can be represented as a finite sum:

$$\int_a^b g(s)\, dW(s) = \sum_{k=0}^{n-1} g(t_k) \left[W(t_{k+1}) - W(t_k) \right]$$

- For a general non-simple process **h(s)** take an approximating sequence of simple processes $h_1(s), h_2(s), \ldots$ s.t.,

$$\int_a^b \mathbf{E}\left[(h_n(s) - h(s))^2 \right] ds \to 0$$

Stochastic Integrals

- It is possible to prove that

$$\int_a^b h_n(s)\, dW(s) \to Z \qquad \text{in } \mathcal{L}^2$$

- Where Z is some r.v. Then define the stochastic integral as follows:

$$\int_a^b h(s)\, dW(s) = \lim_{n\to\infty} \int_a^b h_n(s) dW(s)$$

- Stochastic integrals are often written in "differential form"

$$X(b) - X(a) = \int_a^b h(s)\, dW(s) \qquad \leftrightarrow \qquad dX(s) = h(s)\, dW(s)$$

Stochastic Integrals

- General diffusion processes are often defined through stochastic differential equations:

$$\begin{aligned} dX(s) &= \mu(s, X(s))ds + \sigma(s, X(s))dW(s) \\ X(0) &= x \end{aligned}$$

- The integral representation is:

$$X(t) = x + \int_0^t \mu(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s)$$

- Ito's lemma tells one how the SDE changes under a transformation:

$$\begin{aligned} df(x_t, t) &= \left[\left(\frac{\partial}{\partial t} + \mu(x_t, t)\frac{\partial}{\partial x_t} + \frac{1}{2}\sigma^2(x_t, t)\frac{\partial^2}{\partial x_t^2}\right) f(x_t, t)\right] dt \\ &\quad + \left[\sigma(x_t, t)\frac{\partial}{\partial x_t} f(x_t, t)\right] dW_t \end{aligned}$$

Black-Scholes Differential Equation

- Suppose that an investor follows a self-financing strategy which consists of
  - **$\Delta(S_t,t)$** units of the asset
  - **$\theta_t(S_t,t)$** units in the money-market account;
  - **-1** units in the option in the interval **(t,t+dt]**.
- Let **$V_t(S_t)$** denote the value-process for such a strategy. Then,

$$V(S_t, t) = \Delta(S_t, t)S_t + \theta(S_t, t)M_t - P(S_t, t)$$

- Choose **$\Delta(S_0,0)$** and **$\theta(S_0,0)$** such that **$V(S_0,0)=0$**

$$\begin{aligned} dV(S_t, t) &= d(\Delta(S_t, t)S_t + \theta(S_t, t)M_t) - dP(S_t, t) \\ &= \Delta(S_t, t)dS_t + \theta(S_t, t)dM_t - dP(S_t, t) \end{aligned}$$

Black-Scholes Differential Equation

- Using Ito's Lemma one finds,

$$dV(S_t,t) = \left[\mu S_t \Delta + r M_t \theta - \left(\frac{\partial}{\partial t} + \mu\, S_t \frac{\partial}{\partial S_t} + \frac{1}{2}\sigma^2\, S_t^2\, \frac{\partial^2}{\partial S_t^2}\right) P\right] dt + \left[\sigma S_t \Delta - \sigma S_t \frac{\partial}{\partial S_t} P\right] dW_t$$

- By choosing **$\Delta(S_t,t)$** appropriately, the stochastic term can be removed,

$$\Delta(S_t,t) = \frac{\partial}{\partial S_t} P(S_t,t)$$

- Since the return is non-stochastic, to avoid arbitrage it must grow at the risk-free rate, which leads to the Black-Scholes PDE:

$$\left(\frac{\partial}{\partial t} + r\, S_t \frac{\partial}{\partial S_t} + \frac{\sigma^2}{2}\, S_t^2\, \frac{\partial^2}{\partial S_t^2}\right) P(S_t,t) = r P(S_t,t)$$