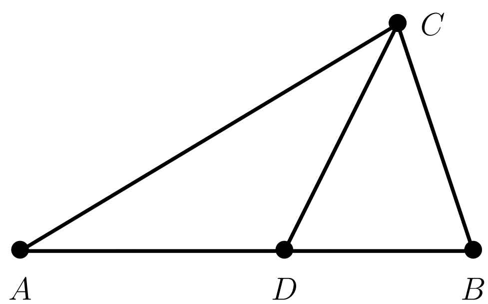
SOAR Math Course | **Homework Three** | Winter, 2003

These homework problems are meant to expand your understanding of what goes on during class. Any you turn in will be graded and returned to you. Answers may or may not be posted on the web, depending on demand.

1. Let $\angle ABC$ be a given angle that is not a straight angle. (That is, the points A, B, and C do not lie on a straight line.) Show that you may choose a line ℓ such that the lines AB and BC are the two lines parallel to ℓ (in the sense of hyperbolic geometry).
   (**Hint:** Look at the bisector of $\angle ABC$. The line ℓ must be perpendicular to this line. Why?)

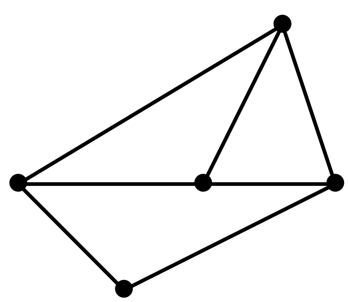
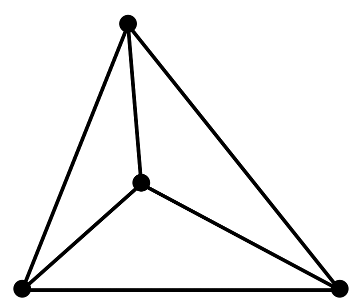
2. In the last two classes we've used the fact that the *defect* of a triangle is additive. This problem explores this fact. (Here $D(\triangle PQR)$ is the *defect* of the triangle $\triangle PQR$. This is related to the angle sum $S(\triangle PQR)$, which is simply the sum of the three angles in a triangle. The defect measures the failure of the angles to sum to what we expect: $D(\triangle PQR) = 180° - S(\triangle PQR)$.

   (a) Here is a triangle $\triangle ABC$ that has been partitioned in to two smaller triangles: $\triangle ACD$ and $\triangle BCD$:

   Show that $D(\triangle ABC) = D(\triangle ACD) + D(\triangle BCD)$.

   (b) Suppose we can break a triangle $\triangle ABC$ into smaller triangles. Explain why $D(\triangle ABC)$ is the sum of the defects of these smaller triangles. In your explanation, be sure to take into account at least the following two cases:

3. For each of the following axiom systems, show whether the system is consistent or not. If it is inconsistent, remove an offending axiom (or two) so that the remaining axioms are consistent. For any consistent system, show consistency with a model (and an explanation of how you created this model).

(a) In this system, undefined terms are "bugs," "potatoes" and "eats."

Axiom 1. There are at least two bugs.

Axiom 2. There are at least four potatoes.

Axiom 3. For every pair of potatoes, at least one bug eats both.

Axiom 4. Every bug eats at least one potato.

Axiom 5. There is at least one potato that no bug eats.

(b) In this system, the undefined terms are "boys" and "greets."

Axiom 1. There are at least three boys.

Axiom 2. If boy A greets boy B and boy B greets boy C, then boy A does not greet boy C.

Axiom 3. Every boy greets himself.

Axiom 4. Exactly one boy is never greeted by anyone other than himself.

(c) Make up your own system and prove its consistency or inconsistency.

4. For each of the axiom systems in the previous problem, determine if any of the axioms are independent. Show that your answer is correct. If, for a particular system, there are no dependent axioms, introduce one and prove that it is dependent.

5. Consider the formal axiom system (introduced in class) that had, as its undefined terms salt, chips, and flavour. The axioms were

   A1 There is at least one salt.

   A2 For any two distinct chips, there is a unique salt that flavours both of them.

   A3 Each salt flavours at least two chips.

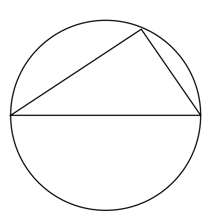
   A4 For each salt, there is at least one chip that it does not flavour.

   (a) Answer the question posed in class: if there are four chips, how many salts are there?

   (b) What if there are five chips?

6. Suppose that the angle of parallelism $\Pi(x)$ is constant (that is, suppose that the angle of parallelism is some fixed angle α, no matter what the distance between the two lines is). Show that $\alpha = 90°$ and that therefore the geometry under consideration is Euclidean.
   (**Hint:** Draw a quadrilateral with two opposite sides on parallel lines and the other two perpendicular to one of the lines. Where are the angles of parallelism in this picture?)

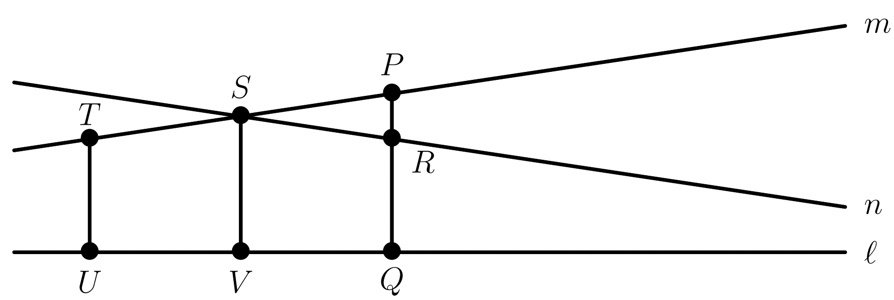
7. Consider a circle with an inscribed triangle. The long side of the triangle is a diameter of the circle.



(a) Prove that the angle opposite the diameter is a right angle in Euclidean geometry.

(b) Prove that the angle opposite the diameter is an acute angle in hyperbolic geometry.

8. Here is a proof we were trying to complete in class. We were proving that, given two parallel lines ℓ and m and a distance x, one can find a point on m so that the distance from that point to ℓ is x. The picture for the proof (explained during the problem) is:



Here is a sketch of the proof we were going through:

(i) Choose a point P on m, drop a perpendicular to ℓ and mark this point Q.

(ii) Assuming (for now) that $x < PQ$, mark a point R on PQ so that $QR = x$. Draw the line n through R parallel to ℓ (but in the opposite direction to m).

(iii) Mark the point where m and n intersect as S. Now find T on m so that $TS = RS$. Drop a perpendicular from T to ℓ; mark this point U.

Your job is to complete this proof.

(a) Complete the above proof. That is, assuming $x = QR < PQ$, show that $TU = QR$ (so $TU = x$, as desired).
(**Hint:** Show that the triangles $\triangle STV$ and $\triangle SRV$ are congruent. Why does $\angle TSV = \angle RSV$?)

(b) Repeat the above prove in the case that $x > PQ$. This means that we must extend QP past m so that $QR = x$ can be greater than QP.