SOAR Math Course Homework Four Winter, 2003

These homework problems are meant to expand your understanding of what goes on during class. Any you turn in will be graded and returned to you. Answers may or may not be posted on the web, depending on demand.

1. In class I mentioned Bolyai's theorem, which says that if two polygonal regions have the same area, then they are "equivalent by finite decomposition." (This means that there is a single finite set of triangles that can tile each of the two polygons.) Explain why this theorem is true for rectangles in Euclidean geometry.

2. Let $\triangle AB\Omega$ be an ideal triangle. Show that the sum of $\angle A$ and $\angle B$ is less than 180°.

3. Let $\triangle AB\Omega$ and $\triangle A'B'\Omega'$ be ideal triangles. Assume that $\angle A = \angle A'$ and $\angle B = \angle B'$. Show that $AB = A'B'$. (This can be thought of as an "Angle-Angle-Angle" congruence theorem for ideal triangles in hyperbolic space. Remember that we already have such a theorem for regular triangles.)

4. Suppose you are given two ideal triangles $\triangle AB\Omega$ and $\triangle A'B'\Omega'$ with $\angle A = \angle A'$.
   (a) Suppose $AB > A'B'$. What can you say about $\angle B$ compared with $\angle B'$?
   (b) Suppose $\angle B > \angle B'$. What can you say about AB compared with $A'B'$?

5. In class we saw that there is not necessarily a line through two distinct ultra-ideal points Γ_1 and Γ_2. Given an ideal point Ω and an ultra-ideal point Γ, is there a line through Ω and Γ? Is it unique? (That is, can there be more than one?)

6. Here is an outline of a proof we skipped in class. We prove that, for any two triangles $\triangle ABC$ and $\triangle A'B'C'$, we have

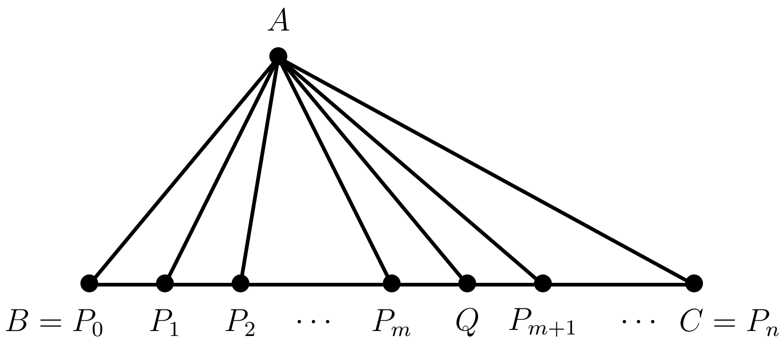
$$\frac{A(\triangle A'B'C')}{A(\triangle ABC)} = \frac{D(\triangle A'B'C')}{D(\triangle ABC)}. \tag{$*$}$$

   In the proof, we'll be using the following fact, often called the *comparison theorem*:

   > Suppose $x > m/n$ if and only if $y > m/n$ for any fraction m/n. Then $x = y$.

   Before we start, you should convince yourself that this is true. (The idea is simply that this means x and y are bounded – above and below – by the same rational numbers. If they were different, there would be a rational number between them. Of course, *proving* this statement is a little bit off-topic, but you should think about it.)

   We begin the proof by noting that if $D(\triangle ABC) = D(\triangle A'B'C')$, then $A(\triangle ABC) = A(\triangle A'B'C')$ by Bolyai's theorem. So we'll assume that $D(\triangle ABC) > D(\triangle A'B'C')$ (if they're different, then one of them must be larger).



Draw the triangle $\triangle ABC$, with the largest angle at A. We choose Q on BC so that $D(\triangle ABQ) = D(\triangle A'B'C')$. Space the points P_1, P_2, and so on so that there are n triangles $\triangle AP_kP_{k+1}$, each with the same defect, namely $D(\triangle AP_kP_{k+1}) = \frac{1}{n}D(\triangle ABC)$. (These points are not presumed to be evenly spaced along BC.) Pick m; we'll show that

$$\frac{m}{n} < \frac{A(\triangle A'B'C')}{A(\triangle ABC)} \quad \text{if and only if} \quad \frac{m}{n} < \frac{D(\triangle A'B'C')}{D(\triangle ABC)}$$

which, by the comparison theorem, is enough to prove equation $(*)$.

(a) Show that

$$\frac{m}{n} < \frac{A(\triangle A'B'C')}{A(\triangle ABC)} \quad \text{if and only if} \quad \frac{m}{n}A(\triangle ABC) < A(\triangle ABQ).$$

(b) Show that for any k (and in particular for $k = m$)

$$A(\triangle ABP_k) < A(\triangle ABQ) \quad \text{if and only if} \quad D(\triangle ABP_k) < D(\triangle ABQ).$$

(c) Explain why $\frac{m}{n}D(\triangle ABC) = D(\triangle ABP_m)$ and $\frac{m}{n}A(\triangle ABC) = A(\triangle ABP_m)$.
Hint: Use Bolyai's theorem which, in this case, you can take to say that if the defects are equal, the areas are equal.

(d) Combine the above to show equation $(*)$.

7. In class we saw that it was possible to have an ideal triangle with area πk^2 (because the defect was $180°$ or, equivalently, the angle sum was 0). Anton asked if it was possible to have similar (but non-congruent) such triangles. Answer Anton's question.
Hint: Consider the models. What does such a triangle look like in your favorite model? Is it possible to have two different such triangles with a common corner Ω?