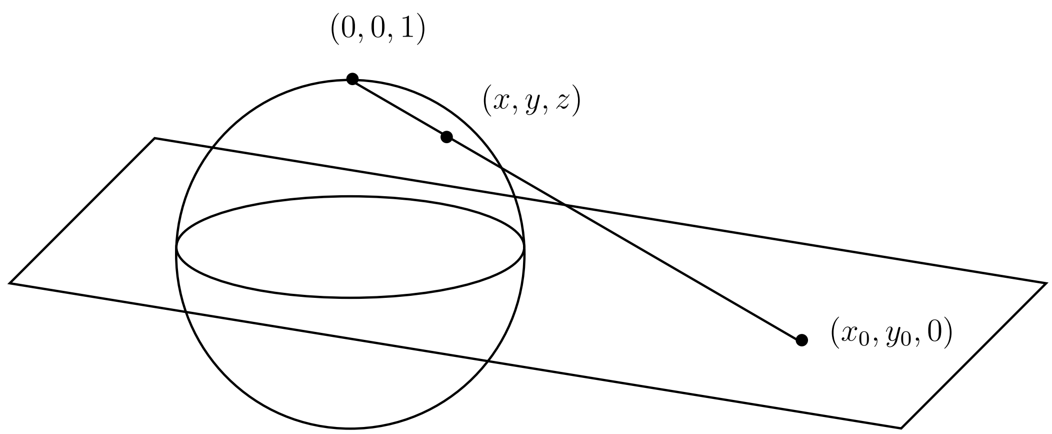
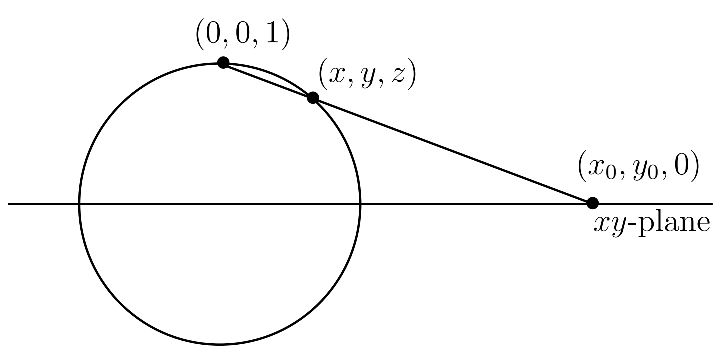
These homework problems are meant to expand your understanding of what goes on during class. Any you turn in will be graded and returned to you. Answers may or may not be posted on the web, depending on demand.

1. Make a model of hyperbolic space, as follows. Print out the page of a tiling of the Euclidean plane by equilateral triangles. Each vertex is surrounded by six triangles. At each of these vertices, cut a slit and add a seventh triangle. (This means that the sheet of triangles can no longer lie flat. This is the whole point: the hyperbolic plane is not flat like the Euclidean plane.) (See also `http://members.tripod.com/professor_tom/hyperbolic/`, where the same sort of construction is done with 7-sided polygons. You could try this, instead, if you like.)

2. This problem shows that great circles on the sphere correspond to circles and lines ("generalized circles") in the Euclidean plane. We're going to define a map from the sphere $x^2 + y^2 + z^2 = 1$ to the Euclidean plane as follows: draw a line from the north pole $(0, 0, 1)$ through the point (x, y, z) on the sphere as in the figure. This will intersect the xy-plane in a point $(x_0, y_0, 0)$ (provided the point (x, y, z) is not the north pole!).



It might also help to view this picture from the side of the xy-plane, as seen below:



(a) Use this second figure and some similar triangles to show that

$$\frac{\sqrt{1 - z^2}}{1 - z} = \sqrt{x_0^2 + y_0^2}.$$

(b) Use similar triangles to show that

$$\frac{x_0}{x} = \frac{y_0}{y} = \frac{1}{1-z}.$$

Hint: Look at the projection of this situation into the three planes: the xy-plane, the xz-plane, and the yz-plane.

(c) From part (a), conclude that

$$x_0^2 + y_0^2 = \frac{1-z^2}{(1-z)^2} = 1 + \frac{2z}{1-z}.$$

A great circle on the sphere $x^2 + y^2 + z^2 = 1$ is the intersection of the sphere with a plane $Ax + By + Cz = 0$ (this is the equation of a plane through the origin – the center of the sphere). We begin by assuming that this great circle does *not* pass through the north pole $(0, 0, 1)$, which means that $C \neq 0$. We can assume that $C = 1$ (just divide through by C and re-name the resulting variables). For the next parts, therefore, assume that $Ax + By + z = 0$.

(d) Using the fact that $Ax + By + z = 0$, show that the equation $x_0^2 + y_0^2 = 1 + \frac{2z}{1-z}$ from part (c) simplifies to

$$x_0^2 + 2Ax_0 + y_0^2 + 2By_0 = 1.$$

(e) The equation from the previous part is a circle. Find the center and radius of this circle. (Your answer will involve A and B.)

Now assume that the great circle on the sphere passes through the north pole $(0, 0, 1)$. In terms of the equation $Ax + By + Cz = 0$, this means that $C = 0$, so $Ax + By = 0$. For this last part, therefore, assume that $Ax + By = 0$.

(f) Using the fact that $Ax+By = 0$, show that the equation from part (b) simplifies to $Ax_0 + By_0 = 0$. This is a line.

(g) At the beginning of this problem, I called circles and lines "generalized circles" on the plane. We've shown that great circles on the sphere correspond to circles in the plane and certain lines in the plane. What lines are these? That is, what lines in the plane correspond to great circles on the sphere?

3. In class we saw that the Cantor set C, obtained from the interval $[0, 1]$ by removing the middle third of each interval repeatedly, has the following properties:

(I) There are no intervals in C. That is, given two points a and b in C, there is a point x *not* in C between a and b (that is, with $a < x < b$).

(II) The length of C is zero. (Well, we saw that the total length of the removed intervals was 1, so that there was no length left over for C.)

Now consider a revised Cantor set, call it K. Obtain K from the unit interval $[0, 1]$ by removing the middle *fifth* (rather than third) of each interval repeatedly. Does K have the same two properties (I) and (II) as C? Explain why your answer is correct.