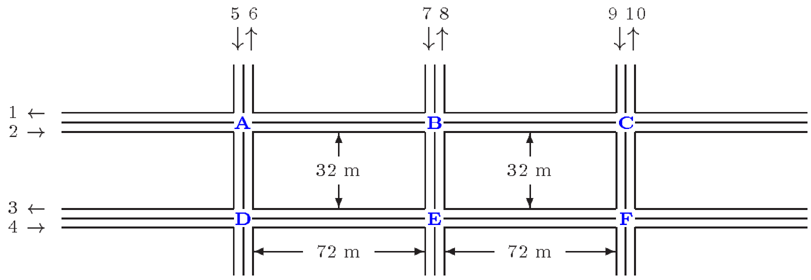
Keep The Traffic Moving!

Welcome to a competition that was held in 1996 by the University of Toronto Mathematics Network. *Although that competition is over, you are still invited to try your hand at exploring* ***mathematics in action*** *in this real life situation!*

The scenario

Everybody knows how frustrating it is to sit at a red light. Worse than that, a light which is red too long can cause cars to back up endlessly, leading to gridlock and other chaos. It is the job of the traffic engineer to time things so that the traffic has the least amount of waiting to do.

In this competition, you are the traffic engineer. You control the timings of the lights at the six intersections shown in the picture below. Each road has two lanes, one in each direction. The lanes are each 4 metres wide (so each road is 8 metres wide), and the spacing between the roads is as shown. For convenience we've numbered the lanes from 1 to 10, indicated their directions, and labelled the lights **A** through **F**.



(If you have Internet access, you can control the lights using a computer simulation, available at `http://www.math.toronto.edu/mathnet/carcompet/simulation.html` on the world-wide-web.)

What You (the Traffic Engineer) Already Know

You have measured the traffic densities $d_1, \ldots, d_{10}$: the number of cars per minute trying to use each lane.

You have also found that the traffic behaviour (the way a car speeds up or slows down when faced with an obstacle in front, such as a red light or another car) follows a certain mathematical model governed by three numbers c, l, and L (the model is described in the section *Details on the Mathematical Model Used* below).

What You Need to Figure Out

This competition has three parts; we encourage you to test your mathematical skills on any or all of them! Parts I and II require you to do some serious mathematical thinking, but in Part III all you have to do is time the lights and see if you can beat the current "high scores" on the computer. So, if you're not in a serious mathematical mood right now, you can skip to part III.

To get the full details on each part, see the following sections. Here's a brief description of them:

1. *How can you use the information you have (the thirteen numbers $d_1, \ldots, d_{10}$, c, l, and L) to decide whether traffic lights are even feasible, or whether a more drastic solution is required, such as widening the roads or building overpasses?*

   In the detailed discussion of this part, we'll give you hints on how basic algebra can be used to come up with a simple set of inequalities that tells whether or not traffic lights would be capable of handling the traffic flow. See if you can figure out this formula. You can test your answer on the computer simulation.

2. *What happens when a long light of cars is waiting at a red light, and the line turns green?* If you've ever been near the back of such a line, you've probably sat in frustration wishing the cars in front would get a move on! But is it only their slowness in reaction?

   In the detailed discussion of this part, we'll explain how ideas of calculus and " differential equations" can be used to answer the question, and challenge you to find out how long it takes for the line to get moving even if everybody's reaction time is instantaneous and there's no limit on the cars' acceleration power. You may be surprised at the answer, and less inclined to blame the cars in front of you next time you're kept waiting! **Note: the mathematics in this part is quite challenging and requires a knowledge of calculus**. But you can still watch what happens on the computer simulation even if you can't solve the problem mathematically.

3. *Find the best light timings for three specific situations.* In the detailed discussion of this part, we give you three different situations (three different sets of numbers for $d_1, \ldots, d_{10}$, c, l, and L). Your job: time the traffic lights in a way that causes the least amount of delay. This part of the competition is a free-for-all: you can use any combination of mathematical reasoning or pure experimentation. The computer simulation lets you enter timing values, watch what happens, and get a score for how well your timings did. It keeps track of the high score, so see if you can beat it and get a new high score!

See the following pages for more information on each of these three competition questions and on the mathematical model used.

Good luck!

Part I: Are Traffic Lights Feasible?

Prerequisites: Basic Algebra.
Mathematical Reasoning/Computer Experimentation balance: primarily MR.

How can you use the information you have (the thirteen numbers $d_1, \ldots, d_{10}$, c, l, and L) to decide whether traffic lights are even feasible, or whether a more drastic solution is required, such as widening the roads or building overpasses? The following steps should help you to answer this question.

Step 1. Traffic lights are feasible only if they're able to let through as many cars per minute as are trying to get through; otherwise there will be an ever-increasing backlog.

When a light is red it lets through no cars at all. When it's green or yellow it lets through cars up to a certain maximum number M per minute (which depends on the speed limit and the way that cars try to space themselves out).

If g is the fraction of time that the light is green or yellow, then the time-averaged maximum number of cars per minute that can pass through the light is Mg (see if you can convince yourself of this). In order to avoid an ever-increasing backlog, *this must not be less than the average number of cars per minute that are trying to use the lane.*

Think about a light where an east-west lane with traffic density d intersects a north-south lane with density d'. If g is the fraction of time it lets through east-west traffic, then $1-g$ is the largest fraction of time for which it can let through north-south traffic. To accommodate all east-west traffic, one must have $Mg \geq d$. To accommodate all north-south traffic, one must also have $M(1-g) \geq d'$. The question is, is there any number g that will make both of those inequalities work? See if you can come up with a condition that M, d, and d' have to satisfy in order for such a g to exist.

If you do this for all six lights, you should find a set of equations or inequalities that the numbers M, $d_1, \ldots, d_{10}$ have to satisfy in order for traffic lights to be feasible.

Step 2. Find a formula for M in terms of the numbers c, l, and L that describe traffic behaviour. To do this you will need to know the details of the mathematical model we are assuming the cars follow (see the section *Details on the Mathematical Model*).

Hint: Suppose cars are travelling at constant speed v km/h (which is $50v/3$ metres per minute) with spacing x metres between the cars. Then the number of cars per minute that will pass a given spot is $N = 50v/(3x)$.

In the section *Details on the Mathematical Model*, we describe how v is related to x, c, l, and L. Use this to eliminate x from the formula for N (still under the assumption that you have a stream of traffic moving at constant speed v). This gives you a value for N that depends of v, c, l, and L.

Ask youself, "thinking of c, l, and L as constants, but allowing v to change, what is the largest value that N can be as different speeds v are considered?". Answering that question tells you what M is (still under the assumption that the traffic is moving at constant speed). You should have a formula now for M in terms of c, l, and L.

Finally, discuss as best you can what, if any, modifications need to be made to the formula if the stream of traffic is accelerating or decelerating.

Step 3. Combine your answers to questions 1 and 2 into a unified procedure for how to decide, if someone tells you the values of $d_1, \ldots, d_{10}$, c, l, and L, whether or not traffic lights are feasible. You may find it helpful to pick some values for these variables, plug them into your formulas, and predict whether or not it's possible to time the lights so as to avoid an ever-increasing backlog. Then enter them into the computer simulation and actually see if can or cannot. Do the results of your experiments confirm your predictions?

Part II: What Happens When a Light Turns Green?

Prerequisites: A good grasp of calculus.
Mathematical Reasoning/Computer Experimentation balance: MR aided by CE.

Imagine a long line of cars stopped at a red light. The light turns green. What happens?

If you've ever been near the back of such a line, you've probably sat in frustration wishing the cars in front would get a move on! But is it only their slowness in reaction?

Let's see how long it would take for the line to get moving even if everybody's reaction time is instantaneous and there's no limit on the cars' acceleration power. You may be surprised at the answer, and less inclined to blame the cars in front of you next time you're kept waiting!

So, we'll assume that, as soon as the light turns green, the lead car immediately jumps up to the speed limit c.

What happens to the car behind it? When they were stopped, the spacing between cars was l (remember, we are assuming the cars' behaviour follows the mathematical model that is described in the section *Details on the Mathematical Model*).

But now that the spacing has started increasing, the second car adjusts its speed proportionally to that spacing. Thus the speed v of the second car is given by an equation of the form $v = mx + b$, where x is the spacing and the constants m and b are determined from c, l, and L. For more details, see the section *Details on the Mathematical Model* on the last page.

$$\begin{array}{ccccc} \boxed{\text{car 2}} & & \boxed{\text{lead car}} & \longrightarrow \textit{direction of travel} \\ \longleftarrow & x & \longrightarrow & \\ B(t) & & A(t) & \end{array}$$

As indicated, let $A(t)$ be the position of the lead car at time t and $B(t)$ the position of the second car, so that $x = A(t) - B(t)$. We want to see what sort of a function $B(t)$ is: how the second car behaves. The easiest way to do this is to take the derivative of x: $x'(t) = A'(t) - B'(t)$. We know that $A'(t)$, the speed of the lead car, is just the constant c, and $B'(t)$, the speed of the second car, is $mx + b$. This gives the following equation for x as a function of t:

$$x'(t) = c - mx(t) - b$$

This is probably not an equation you've learned how to solve, since it involves both the unknown function $x(t)$ *and* its derivative. So, here's the way to solve it. Bring $x(t)$ to the left-hand side and multiply both sides by e^{mt} to get

$$e^{mt}x'(t) + me^{mt}x(t) = (c - b)e^{mt}$$

The point of multiplying by e^{mt} is that now the left-hand side is the derivative of a recognizable function: it's the derivative of $e^{mt}x(t)$. We therefore have

$$\frac{d}{dt}e^{mt}x(t) = (c - b)e^{mt}.$$

Integrate both sides to get $x(t)e^{mt} = (c - b)e^{mt}/m$, then divide both sides by e^{mt} to get

$$x(t) = \frac{c - b}{m} + Ce^{-mt}.$$

Solve for C using the fact that, at time $t = 0$, the spacing is $x(0) = l$.

Now that you have a formula for $x(t)$, you should be able to find a formula for $B(t)$, the position of the second car. (You could have found it directly rather than going through finding $x(t)$ first, but the integration is slightly more tedious).

Continue in this way: find formulas for the motion of the third, fourth, and subsequent cars. Much of the perceived annoyance when you're in a car is how long it takes for the traffic in front of you to start moving after a light turns green. Have you ever wished those drivers would be a little faster at accelerating? Well, here, remember, we are assuming that the cars are accelerating as fast as possible. And yet, if you correctly find the formulas, you should discover that it *still* takes quite a long time for the fifth or sixth car in line to get up to speed.

Try to answer the following questions: how long does it take the nth car to achieve (a) 10% of full speed, (b) 50% of full speed, (c) 90% of full speed.

You probably can't solve the whole problem exactly, but use a combination of exact formulae plus experimental evidence from the computer simulation on the World-Wide Web (see page 1 for the address). After entering that address into your web browser, read the simulation introduction if you want, then go to the "Green Light Start-Up" section and press "Start Simulation". Cars will line up at a red light at B for 120 seconds. Then the light turns green and you can watch what happens, combining this evidence with your mathematical calculations. Also try it with a non-zero value for the lead car's acceleration time, and see how much that affects things.

Additional questions to try to answer: Under our assumptions, will a car approaching a red light ever stop, or just keep moving imperceptibly slowly? Will the second and subsequent cars in line when the light turns green ever make it up to full speed c? What does this say about the realism of the assumptions we've made? And, even if our assumptions aren't completely accurate, do you think the inaccuracies are significant or minor? Finally, can you spot any "little white lies" that I have told in order to keep things simple?

Part III: Find the Best Light Timings

Prerequisites: None.
Mathematical Reasoning/Computer Experimentation balance: primarily CE.

Below are three different situations (three different sets of numbers for d_1, …, d_{10}, c, l, and L). Your job: time the traffic lights in a way that causes the least amount of delay. This part of the competition is a free-for-all: you can use any combination of mathematical reasoning or pure experimentation. The computer simulation lets you enter timing values, watch what happens, and get a score for how well your timings did. It keeps track of the high score, so see if you can beat it and get a new high score!

The Three Scenarios

In each scenario, $c = 40$, $l = 4$, and $L = 20$; the only difference is in the values for the densities d_1, …, d_{10}: the number of cars per minute trying to use each lane.

Scenario 1. $(d_1, \ldots, d_{10}) = (10, 5, 5, 10, 15, 10, 8, 8, 5, 10)$

Scenario 2. $(d_1, \ldots, d_{10}) = (5, 15, 2, 15, 2, 10, 1, 8, 10, 5)$

Scenario 3. $(d_1, \ldots, d_{10}) = (7, 20, 15, 15, 12, 6, 10, 10, 2, 12)$

How to Time the Lights

To time the lights, you need to pick, for each light, two numbers: how many seconds it lets east-west traffic through, and how many seconds it lets north-south traffic through. Let's call these numbers A_{ew} and A_{ns} for light A, B_{ew} and B_{ns} for light B, and so on.

You might also get more efficient results by making the lights turn green at different times, so that a lane with heavy traffic can encounter all green lights (for instance, making the second light in the lane turn green just before cars released by the first light reach it).

The scenario begins just as light A starts its east-west green cycle. To stagger the lights, you can specify how far into the scenario lights B through F begin their east-west green cycle. Let's call these numbers B_0 through F_0. For example, setting $B_0 = 15$ means that light B will begin an east-west green cycle 15 seconds after the start of the scenario.

That gives a total of 17 numbers: A_{ew}, A_{ns}, …, F_{ew}, F_{ns}, B_0, …, F_0. Your job is to choose values of these numbers which will lead to the smallest average waiting time for cars over a 10-minute period, for the scenario you chose.

You may want to refer back to the map in the introduction to see where each of the ten lanes is located relative to the six lights, think about the density values given, and try to think about what kind of light timings might be most efficient for the given density values.

To try out your timing values on the computer simulation (on the World-Wide Web at the address `http://www.math.toronto.edu/mathnet/carcompet/simulation.html`), go to the "Predefined Scenarios" section, enter your timing values, and press "Start Simulation". (The yellow stage of the light is set automatically at 5 seconds. For instance, if you entered "A_{ew}= 30", east-west traffic would see light A green for 25 seconds and yellow for 5 seconds).

You will see a score telling you how efficient your timings were. You will also see the best score that anybody else has achieved; see if you can beat that high score!

Details on the Mathematical Model

We're assuming that traffic behaves according to the following model:

- As long as there's no obstruction (another car or a red light) in front of a car, it will travel at the speed limit of c kilometres per hour.
- If a car gets within a distance of L metres from an obstruction (where, to avoid having to think about the car length as a separate factor, we measure distances from the back of the car), it will reduce its speed proportionally to that distance.
- If a car gets within a distance of l metres from an obstruction, it will stop altogether.

Is this model realistic? Not really; it does not take into account drivers' reaction times, limits to how fast a car can brake or accelerate, and other factors. But, although it's not completely realistic, the inadequacies of the model don't make a whole lot of difference to the basic nature of the answers you get. Working with a more realistic model requires much, much harder mathematical calculations, and the final answers aren't all that much different! So, we'll stick with this simple model.

Let's be a little more precise about how this model works. Let x denote the distance in metres between two cars (measured from the back of one to the back of another), or between the back of one car and a red light up ahead.

This is indicated below:

$\boxed{\text{car}}$ $\qquad$ $\boxed{}$ red light, or back of another car $\longrightarrow$ *direction of travel*

$\longleftarrow x \longrightarrow$

The speed v of the car is:

$$\begin{array}{cl} c, & \text{if } x \geq L \\ \text{between 0 and } c, & \text{if } l < x < L \\ 0, & \text{if } x \leq l. \end{array}$$

In the second case, $l < x < L$, we're assuming the car changes speed proportionally to changes in distance, so that v is given by a "linear" function of x (the graph of v as a function of x is a stratight line). This means that $v = mx + b$ for two constants m and b. These constants can be determined from the two conditions

1. $v = 0$ when $x = l$
2. $v = c$ when $x = L$;

from these, you can solve for m and b in terms of the three fundamental numbers c, l, and L.

In summary, then, we're assuming that the speed of a car is given by the following function of the distance x to the closest obstacle in front of it:

$$\begin{array}{cl} v = c, & \text{if } x \geq L \\ v = mx + b, & \text{if } l < x < L \\ v = 0, & \text{if } x \leq l. \end{array}$$

where m and b can be expressed in terms of c, l, and L (which we leave for you to do). These three numbers, therefore, completely determine the traffic behaviour (according to our model).