SOLUTIONS TO THE PROBLEMS.

Problem 1. Let B denote the whole $20 \times 20 \times 20$ block and U the unit cube of the problem, whose assigned number is 10. Let S_1, S_2 and S_3 denote the three $1 \times 20 \times 20$ slices of B, parallel to the faces of B and containing U. Consider the three columns of B, containing U and parallel to the edges of B. These are the pairwise intersections of S_1, S_2 and S_3. Let us denote these three columns by $E_{12} = S_1 \cap S_2$, $E_{13} = S_1 \cap S_3$ and $E_{23} = S_2 \cap S_3$. Let $T = S_1 \cup S_2 \cup S_3$ and let C be the set whose sum of the numbers we want to find out: $C = B \setminus T$. Let us denote the sum of the numbers of unit cubes in each of these sets by a corresponding small letter, so that b is the sum of the numbers of cubes in B, s_1 is the sum of the numbers of cubes in S_1, etc.

Each $1 \times 20 \times 20$ slice S parallel to a face of B can be broken up into 20 columns, each of size $1 \times 1 \times 20$. The sum of the numbers in each column is 1 by assumption, hence the sum of all the numbers in S is $s = 20 \times 1 = 20$; in particular, the sum s is the same for all such slices S. Now, 20 such slices make up all of B, so the sum total of all the numbers in B is $b = 20 \times 20 = 400$. We want to find the number c. We have $c = b - t = 400 - t$. It remains to find t. To do that, consider the sum $s_1 + s_2 + s_3$. The difference between this sum and t is that in $s_1 + s_2 + s_3$, each of the columns E_{12}, E_{13} and E_{23} has been counted twice. To compensate for that, consider the number $s_1 + s_2 + s_3 - e_{12} - e_{13} - e_{23}$. In this expression, the number of each unit cube in T has been counted once, except for u, which has been counted $1+1+1-1-1-1 = 0$ times. Making the final correction to account for that, we obtain $t = s_1 + s_2 + s_3 - e_{12} - e_{13} - e_{23} + u = 20 + 20 + 20 - 1 - 1 - 1 + 10 = 67$, so $c = b - t = 400 - 67 = 333$ is the desired number.

Problem 2. Let n be an integer such that n^2 has 9 as the units digit and 0 as the tens digit. Then n^2 is odd, so n must also be odd. Let us investigate residues modulo 8.

Write $n = 2k + 1$ for some integer k. Then $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 4k(k+1) + 1$. Now, $k(k+1)$ is even for all integer k (since one of k and $k+1$ must be even). Hence $4k(k+1)$ is divisible by 8 and $n^2 \equiv 1 \bmod 8$ for all odd integers n.

Since $100 \equiv 4 \bmod 8$, any odd multiple of 100 is congruent to 4 mod 8. We have $9 \equiv 1 \bmod 8$. Let d denote the hundreds digit of n^2. If d were odd, we would have $n^2 \equiv 4d + 9 \equiv 4 + 1 \equiv 5 \bmod 8$, which is impossible by the above. This proves that d is even, as desired.

Problem 3. Let us denote the twelve statements by the respective candidates by S_1, S_2 and so on until S_{12}.

Lemma.

(1) *If the number of lies told before S_i is greater than i then the number of lies*

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told before S_{i+1} is greater than $i+1$.

(2) *If the number of lies told before S_i is less than or equal to i then the number of lies told before S_{i+1} is less than $i+1$.*

Proof. (1) In this case S_i is a lie. When S_i is said the number of lies increases by 1 and thus becomes greater than $i+1$.

(2) First, suppose the number of lies told before S_i was strictly less than i. When S_i is said, the number of lies goes up by one, and thus the resulting number of lies is strictly less than $i+1$.

If the number of lies tolde before S_i was exactly i then S_i is true so the number of lies told before S_{i+1} is $i < i+1$. □

Corollary. *The statement S_{i+1} is a lie for $1 \leq i \leq 11$.*

Thus S_2, S_3, ... S_{12} are all lies. Since at least of the S_i was true by assumption, S_1 must be true. Thus exactly 1 lie was told before S_i, exactly 11 lies after S_1, and S_1 itself was true, the total number of lies is $1 + (12 - 1) = 12$.

Problem 4. We proceed by induction on the number $n + m$. For convenience in setting up the induction, we will allow one of n or m to be zero (the statement of the problem is true also in this case).

Base of the induction. Assume that one of m and n is even and the other is odd (this includes the case $n + m = 1$, in which case one of n and m is zero, the other 1). In this situation, the infinite chessboard with the usual black and white pattern will do. Indeed, say m is even and n is odd. Then, if the crocodile starts on a black square, after moving m squares in one direction it lands on a black square. Moving n squares in another direction, it lands on a white square, as desired.

The induction step. Fix m and n and assume that the statement is known for all the smaller values of $m+n$. The case when one of m and n is even and the other odd is already solved above. It remains to consider the cases when either both m and n are odd or they are both even.

First, suppose m and n are both odd. Let us number the rows and columns of our chessboard by integers from $-\infty$ to $+\infty$. Paint all the even columns white and all the odd columns black. When the crocodile moves an odd number of squares vertically, it stays on the square of the same colour, and when it moves an odd number of squares horizontally, it lands on a square of a different colour than the one it started on.

Finally, suppose both m and n are even. Divide all the squares of the chessboard into the following four sets. Let C_1 be the set of squares such that both row and column number is even, C_2 the squares such that both row and column are odd, C_3 the squares in an even row and odd column and C_4 the squares in an odd row and even column. Since both m and n are even, the crocodile which starts out in one of the sets C_i will always remain in the same set C_i. Thus it suffices to solve the problem for each of the sets C_i separately. Clearly the problem is the same for each of the four sets, so it is enough to solve it for C_1. Now, C_1 itself can be identified with an infinite chessboard. Renumber the rows of this new chessboard so that they are again numbered by all the integers from $-\infty$ to $+\infty$, instead of just the even ones. We see that in this new chessboard our piece becomes a mere $\left(\frac{n}{2}, \frac{m}{2}\right)$-crocodile, so the desired painting of the chessboard exists by the induction assumption.

Problem 5. Let n and $n+1$ be two consequtive positive integers and $2m$, $2m+2$ two consequtive even positive integers. Suppose that $n(n+1) = 2m(2m+2)$. We have

$$n^2 + n = 4m^2 + 4m > 4m^2 + 2m, \tag{1}$$

hence

$$2m < n. \tag{2}$$

By (2), $n \geq 2m+1$, so $n^2 + n \geq (2m+1)^2 + 2m + 1 = 4m^2 + 4m + 1 + 2m + 1 = 4m^2 + 6m + 2 > 4m^2 + 4m$. This is a contradiction, so this situation is impossible.

Problem 6. Let us consider the usual unit cube in the three-dimensional Eucledian space, whose eight vertices are precisely the set of all points, all of whose coordinates are either 0 or 1. Now, no two vertices whose sum of coordinates is even are joined to each other by an edge. Similarly for vertices whose sum of coordinates is odd. On the other hand, every vertex whose sum of coordinates is even is joined by edges to three vertices with odd sum of coordinates and vice-versa. Now take four distinct prime numbers, say, 2,3,5 and 7, and place them in the vertices with even sum of coordinates. Say, we place 2 at (0,0,0), 3 at (0,1,1), 5 at (1,0,1) and 7 at (1,1,0). Finally, in every vertex v whose sum of coordinates is odd we place the product of the three prime numbers from the three vertices connected to v by an edge. Specifically, in the case at hand we place 70 at (1,0,0), 42 at (0,1,0), 30 at (0,0,1) and 105 at (1,1,1).

We claim that such a cube satisfies the requirements of the problem. First, consider a pair of numbers connected by an edge. Then one of these numbers a, is assigned to a vertex with an even sum of coordinates and the other, b, to a vertex with an odd sum of coordinates. By construction, a is prime and $a \mid b$, as desired.

In two vertices with even sum of coordinates we have two distinct prime numbers, so clearly they do not divide each other. In two vertices with odd sum of coordinates we have products of two distinct sets of three primes, so they cannot divide each other. Finally, given a vertex v with even sum of coordinates and the vertex v' with odd sum of coordinates lying on the same main diagonal as v', we have a prime number p at v and a product of three primes different from p in v', so, again, they do not divide each other. This provides the desired example.

Problem 7. Let us colour our 16×16 square into black and white squares, as in the usual chessboard. The neighbours of every black square are all white and the neighbours of every white square are all black. Thus if we can find the sum of the numbers of all black squares, the sum of all white squares will be the same by symmetry and the problem will be solved.

Let us denote each square by its row and column number. The idea is to divide the eight black squares into several disjoint sets, in such a way that each set is a complete set of neighbours of some white square.

Namely, let A be the set consisting of (1,2) and (2,1): these are all the neighbours of (1,1). Let $B = \{(1,4),(2,3),(3,4)\}$: these are all the neighbours of (2,4). Let $C = \{(4,1),(3,2),(4,3)\}$: these are all the neighbours of (4,2). The sum of all the numbers in A is 1 by assumption, similarly for B and C. Thus the sum of all the numbers on the black squares is 3, hence the sum on the white squares is also three by symmetry and the total sum is 6.

Problem 8. The idea is to consider the smallest eight among the 100 numbers. First, we prove a lemma which will be used in both parts (a) and (b).

Lemma. *Consider any collection of 100 numbers (which may or may not be distinct). Let $a_1 \le a_2 \le a_3 \le a_4 \le a_5 \le a_6 \le a_7 \le a_8$ be the smallest eight numbers, arranged in the increasing order. Assume that there exist nine numbers $b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9$ among the given hundred whose arithmetic mean equals the arithmetic mean of $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8$. Then all the a_i and all the b_j are equal to each other.*

Proof. Arrange the b_i also in the increasing order. By the choice of the numbers a_i, we have

$$a_i \le b_i \tag{3}$$

for all i from 1 to 8. We also know that

$$b_i \le b_9 \tag{4}$$

for all i from 1 to 8. Adding (4) to (3) multiplied by 8, we deduce that

$$9a_i \le 8b_i + b_9 \quad \text{for } i \in \{1, 2, \dots, 8\}. \tag{5}$$

Letting i range from 1 to 8 and adding up the resulting eight inequalities (5), we obtain $9(a_1+a_2+a_3+a_4+a_5+a_6+a_7+a_8) \le 8(b_1+b_2+b_3+b_4+b_5+b_6+b_7+b_8)+8b_9$, so

$$\frac{a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8}{8} \le \frac{b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 + b_8 + b_9}{9}. \tag{6}$$

By assumption, (6) is, in fact, an equality. Hence (3) and (4) must also be equalities for all i. Thus all of the a_i and b_i are equal to b_9. $\square$

Proof of (a). If of all the 100 numbers are diffierent then the nine smallest numbers cannot all be equal so, by the Lemma, the arithmetic mean of the smallest eight numbers $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8$ cannot equal the arithmetic mean of any 9 of the written numbers.

(b) We will give a proof assuming all the 100 numbers are rational. I still do not know how to do the general case (that of real numbers) without using linear algebra.

We give a proof by contradiction. Start with 100 rational numbers, not all equal, satisfying the assumptions of the problem. In particular, the arithmetic mean of the smallest eight numbers equals the arithmetic mean of some nine numbers, so by the Lemma the smallest nine numbers are all equal to each other.

Multiplying all the numbers by the same constant or adding the same constant to all the numbers does not change the problem. Adding a suitable constant to all the numbers, we may assume that the nine smallest numbers are all equal to 0. Multiplying by the lowest common denominator of all the numbers, we may assume that all the numbers are integers.

We want to show all the numbers are equal to 0. Suppose not. Dividing by the greatest common divisor of all the numbers, we may assume that no integer greater than one divides all the numbers. In particular, not all 100 numbers are divisible by 8. Let a'_8 be a number not divisible by 8. Let

$$a'_1 = a'_2 = a'_3 = a'_4 = a'_5 = a'_6 = a'_7 = 0.$$

Let $b'_1, b'_2, b'_3, b'_4, b'_5, b'_6, b'_7, b'_8, b'_9$ be nine numbers such that

$$\frac{a'_1 + a'_2 + a'_3 + a'_4 + a'_5 + a'_6 + a'_7 + a'_8}{8} = \frac{b'_1 + b'_2 + b'_3 + b'_4 + b'_5 + b'_6 + b'_7 + b'_8 + b'_9}{9}. \tag{7}$$

Then

$$9(a'_1 + a'_2 + a'_3 + a'_4 + a'_5 + a'_6 + a'_7 + a'_8) = 8(b'_1 + b'_2 + b'_3 + b'_4 + b'_5 + b'_6 + b'_7 + b'_8 + b'_9). \tag{8}$$

Since $b'_1, b'_2, b'_3, b'_4, b'_5, b'_6, b'_7, b'_8, b'_9$ are all integers, $a'_1 + a'_2 + a'_3 + a'_4 + a'_5 + a'_6 + a'_7 + a'_8 = a'_8$ is divisible by 8, which is a contradiction.

Problem 9. We will describe a solution for both (a) and (b) simultaneously. The idea of using binary to number the coins was suggested by SIMMER participants during the discussion.

In Case (a), number the coins from 0 to 31. In Case (b), number the coins from 0 to 10 and from 21 to 31. Write all the numbers out in binary. All the resulting numbers have at most five digits; let us refer to them as "five digit numbers". In both cases (a) and (b), the resulting set S of five digit binary numbers can be divided into pairs (x, y) such that $x + y = 11111$ (in binary). This means that the i-th digit of x is 0 if and only if the i-th digit of y is 1, for $i \in \{1, 2, 3, 4, 5\}$. Thus for each $i \in \{1, 2, 3, 4, 5\}$ there are as many numbers in S with the i-th digit 0 as there are numbers with the i-th digit 1.

Now consider the following four weighing operations. Let the operation number i consist of weighing all the coins with i-th digit 1 against all the coins with i-th digit 0. If in one of the operations the two sets of coins have the same weight, there is nothing more to do. Suppose in each of the four operations the two groups of coins are not equal. This mean, in each case, that the counterfeit coins cannot be divided between the two groups being weighed: both counterfeit coins must be on the same side in each weighing (we do not know which one).

In the first weighing operation, this means that both counterfeit coins have the same first digit. Similarly, the second, third and fourth weighings show that they must have the same second, third and fourth digits. Therefore the two counterfeit coins must differ from each other by their last digit. Hence the set of coins whose numbers have 0 as their last digit has the same weight as those having 1 as their last digit and we are done.

Problem 10. Without loss of generality, assume that $a \leq b \leq c \leq d$. We cannot have $a = b = c = d$ (for that would imply $l.c.m(a, b, c, d) = d$). Hence

$$d < a + b + c + d < 4d. \tag{9}$$

Since $d \mid (a + b + c + d)$ by assumption, (9) leaves two possibilities:

$$a + b + c + d = 3d \tag{10}$$

or

$$a + b + c + d = 2d. \tag{11}$$

If (10) holds, $3 \mid 3d = l.c.m(a, b, c, d) \mid abcd$ and we are done. From now on we will assume that (11) holds. Then

$$a + b + c = d. \tag{12}$$

If we had $a = b = c$ then $d = 3a = 3b = 3c$, so d is divisible by a, b and c and

$$l.c.m(a, b, c, d) = d, \tag{13}$$

contradiction. Thus at least one of a and b must be strictly less than c. Then $d = a + b + c < 3c$, so

$$2c < 2d < 6c. \tag{14}$$

Since $c \mid l.c.m(a, b, c, d) = 2d$, inequality (14) leaves the possibilities $2d = 3c$, $2d = 5c$ or

$$2d = 4c. \tag{15}$$

In the first two cases, $3 \mid d$ or $5 \mid d$ and we are done. Assume that (15) holds. Then $d = 2c$. By (12), we have $a + b = c$. Again, the case $a = b$ is impossible since then a, b and c divide d and we get the contradiction (13). Thus $a < b$ so $2c > 2b > c$. Then

$$8b > 4c = 2d > 4b. \tag{16}$$

Since $b \mid 2d$, (16) leaves the possibilities $5b = 2d$, $6b = 2d$ or

$$7b = 2d. \tag{17}$$

In the first two cases d is divisible by 3 or by 5 and we are done. Assume that (17) holds. Then $4(a+b) = 4c = 2d = 7b$, so $4a = 3b$. Thus $3 \mid a$ and the proof is complete.

Remark. In fact, using this reasoning it is easy to give a complete list of all the quadruples a, b, c, d, satisfying the conditions of the problem. Multiplying all of a, b, c, d by the same integer does not change the problem, so we may assume that $g.c.d(a, b, c, d) = 1$. Under this assumption, the possible answers are (1,3,4,4), (2,3,3,4), (1,1,4,6), (1,6,14,21), (1,3,8,12), (1,2,6,9), (2,3,10,15), (1,2,2,5) and (1,4,5,10).

Problem 11. The idea, as in many optimization problems, is to look for as symmetric a situation as possible. First, we give an example with 2000 students. Number the problems from 1 to 6. Our plan is as follows. Every student will solve exactly three problems. Consider the set of all triples of distinct integers between 1 and 6. There are $\binom{6}{3} = \frac{6\cdot4\cdot5}{3!} = 20$ of them. We will choose a subset S consisting of ten triples with the following properties:

(1) Each integer from 1 to 6 appears in exactly five triples in S (this property is needed to ensure that each problem got solved by exactly 1000 students)
(2) the union of any two triples from S does not equal $\{1, 2, 3, 4, 5, 6\}$ (this property is needed to ensure that no two students together have solved all the six problems).

Then we will divide our 2000 students into ten groups of 200 students each and assign to each group a triple from S. The problems solved by the students in each group will be the problems whose numbers appear in the corresponding triple. This will provide the desired example.

It remains to describe a set of ten triples satisfying (1) and (2). For instance, we can take the set of all triples whose sum is congruent to 0, 1 or 4 mod 6:

$$S = \{(123), (246), (156), (345), (124), (235), (136), (145), (256), (346)\}.$$

Since $1+2+3+4+5+6 = 21 \equiv 3 \mid \text{mod } 6$, our conditions on the residue of the sum ensure that no two triples in S combined make up $\{1, 2, 3, 4, 5, 6\}$. This completes the construction of the example.

Now we must show that 2000 is the smallest number of students for which an example is possible. Indeed, let x denote the total number of students. We will show that $x \geq 2000$. By assumption, no student solved all six problems. There are three cases to consider.

Case 1. There exists a student A who solved exactly five problems. Without loss of generality, assume that A solved the problems 1,2,3,4,5. Let B be a student who solved problem 6 (by assumption, there are 1000 such students). Then A and B

together solved all the problems, which contradicts the assumptions of the problem. Hence Case 1 is imposiible.

Case 2. There exists a student A who solved exactly four problems. Without loss of generality, assume that A solved the problems 1,2,3,4. Let B_5 denote the set of all the students who solved problem 5. Let B_6 denote the set of all the students who solved problem 6. Each of the sets B_5 and B_6 contains exactly 1000 students by assumption. We have $A \notin B_6 \cup B_5$. Furthermore, if there existed a student $C \in B_5 \cap B_6$ then together A and C would have solved all the problems, which is impossible. Thus $B_5 \cap B_6 = \emptyset$. Then the total number of students in $B_5 \cup B_6 \cup A$ is $1000 + 1000 + 1 = 2001 > 2000$, so $x > 2000$.

Case 3. No student solved more than three problems. Consider the set of all ordered pairs of the form (C, i) where C is a student and $i \in \{1, 2, 3, 4, 5, 6\}$ is a problem solved by student C. Since every one of the six problems was solved by exactly 1000 students, the total number of such pairs is exactly $6 \times 1000 = 6000$. On the other hand, since each one of the x students solved at most three problems, the total number of pairs is at most $3x$. Thus $6000 \leq 2x$ so $x \geq 2000$ as desired. $\square$