

Split “P” Soup
Modular Arithmetic Games
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Here are a couple of two-player games that can be played using modular arithmetic.

The Modular Addition Game. This is the simpler of the two games. Before starting to play, one of the players picks a positive integer M , which will be the modulus for all the modular arithmetic in the game to be played. The other player then gets to decide who goes first. A running total T is set to zero.

At each player’s turn, he/she chooses any integer n and adds it to the running total T . The player loses if either the running total T is a multiple of M , or the running total T is congruent, modulo M , to a previous running total in that game.

Example. Alice and Bob play the Modular Addition Game. Alice chooses the modulus $M = 5$, and Bob decides to go first. Bob chooses the integer 7, so the first running total is 7. Alice adds the integer -88 , obtaining the new running total -81 . Bob adds the integer 102 to obtain the new running total 21. Alice adds -23 to obtain the total -2 . Bob adds 13 to obtain the total 11.

At the end of Bob’s last turn, the sequence of running totals has been 7, -81 , 21, -2 , 11. Bob loses because $11 \equiv 21 \pmod{5}$ —in other words, 11 and 21 leave the same remainder, 1, when divided by 5. Bob would also have lost by adding 12 instead of 13, since then the running total would have been 10, which is a multiple of the modulus 5. (In fact, you can check that Bob would have lost no matter what he did.)

The Modular Multiplication Game. This game is very similar to the Modular Addition Game, but the strategy turns out to be somewhat more complex. Again one of the players picks a positive integer M , which will be the modulus for the game to be played; the other player then gets to decide who goes first. This time, the running total T is set to one.

At each player’s turn, he/she chooses any integer n and *multiplies* the running total T by n . Once again, the player loses if either the running total T is a multiple of M , or the running total T congruent, modulo M , to a previous running total in that game (not counting the initial value of 1).

Example. Alice and Bob play the Modular Multiplication Game. Alice chooses the modulus $M = 15$, and Bob decides that Alice goes first. Alice chooses the integer 2, so that the first running total is 2. Bob multiplies by 6 to obtain 12. Alice multiplies by 2 to obtain 24. Bob multiplies by 9 to obtain 216. Alice multiplies by 3 to obtain 648. Bob multiplies by -5 to obtain -3240 .

At the end of Bob’s last turn, the sequence of running totals has been 2, 12, 24, 216, 648, -3240 . Bob loses because -3240 is a multiple of the modulus 15. Bob would also have lost by multiplying by -7 rather than -5 , since then the running total would have been -4536 , and $-4536 \equiv 24 \pmod{15}$ —in other words, -4536 and 24 leave the same remainder, 9, when divided by 15.

In both of these games, the player who chooses who plays first can always win if he/she plays perfectly! Can you figure out what the winning strategy is ... before it’s too late?!