Solutions to Problems 1–10.

Problem 1. The idea is to compare the binomial expansion of

$$99^{50} = (100-1)^{50}$$

with that of

$$101^{50} = (100+1)^{50}.$$

We have

(1)

$$(100+1)^{50} = \sum_{k=0}^{50} \binom{50}{k} 100^{50-k} = \sum_{k=0}^{50} \frac{50 \cdot 49 \cdot \ldots \cdot (50-k+1)}{k!} 100^{50-k} =$$
$$= 100^{50} + 50 \cdot 100^{49} + \cdots + 50 \cdot 100 + 1$$

(2)

$$(100-1)^{50} = \sum_{k=0}^{50} \binom{50}{k} (-1)^k 100^{50-k} =$$
$$= \sum_{k=0}^{50} \frac{50 \cdot 49 \cdot \ldots \cdot (50-k+1)}{k!} (-1)^k 100^{50-k} =$$
$$= 100^{50} - 50 \cdot 100^{49} + \cdots - 50 \cdot 100 + 1.$$

The terms corresponding to the even values of k are the same in (1) and (2), while those corresponding to the odd values of k differ by sign. Subtract (2) from (1). The terms with k even will cancel, while those with k odd will be double of what they were in (1):

$$(100+1)^{50} - (100-1)^{50} = 2 \cdot \sum_{l=1}^{25} \frac{50 \cdot 49 \cdot \ldots \cdot (50-2l+2)}{(2l-1)!} 100^{50-2l+1} =$$
$$= 2 \cdot (50 \cdot 100^{49} + \text{ a sum of positive terms}) > 100^{50}.$$

This shows that $101^{50} - 99^{50} > 100^{50}$, so $99^{50} + 100^{50} < 101^{50}$.

Problem 2. (a) To say that two rooks cannot take each other is the same as to say that they are not in the same row nor in the same column. Thus to place 8 rooks so that no two can take each other means to place them so that no two lie in the same row or in the same column. Since there are 8 rooks, 8 rows and 8 columns, there must be exactly one rook per row and exactly one rook per column. There are 8 ways of placing a rook in the first column. For each one of those 8 possibilities of placing the first rook, there are 7 ways of placing the second rook in the second column such that rook 1 and rook 2 do not lie on the same row. For each of the $7 \cdot 8$ ways of placing the first two rooks, there are six ways of placing the third rook in the third column so that rook 3 does not lie on the same row with either of the rooks 1 and 2, and so on. Finally, there are $8 \cdot 7 \cdot 5 \cdot 6 \cdot 4 \cdot 3 \cdot 2$ ways of placing the first seven rooks, and then the position of the last rook is uniquely determined by the first seven. Thus the answer is $8! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8$.

(b) We must place 4 rooks on the 8×8 chessboard so that no two are in the same row nor in the same column. In total, four columns will contain a rook and four others will not. First, let us choose the set of 4 columns in which rooks will be placed. There are $\binom{8}{4} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4!} = 70$ choices for this. Make one such choice, and let C_1, C_2, C_3 and C_4 denote the chosen columns. Now, for each choice of C_1, C_2, C_3, C_4, there are 8 ways to choose one rook from column C_1, for each of those there are 7 ways to choose a rook in C_2, then 6 ways to choose a rook in C_3 and 5 ways to choose a rook in C_4. Final answer: $\binom{8}{4} \cdot 8 \cdot 7 \cdot 6 \cdot 5$ ways.

Problem 2. The idea is to use the transformation T of the plane. given by

$$(x, y) \to \left(\frac{x}{\sqrt{3}}, y\right), \qquad (3)$$

which maps the ellipse onto the circle given by $3x'^2 + 3y^2 = 3$, or $x'^2 + y^2 = 1$, where $x' = \frac{x}{\sqrt{3}}$ is the new coordinate. The line $x = \frac{\sqrt{6}}{2}$ maps to the line $x' = \frac{\sqrt{2}}{2}$. The two intersection points of $x = \frac{\sqrt{6}}{2}$ with the ellipse are mapped to the points $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ on the circle. The points $(0,0)$, $\left(\frac{\sqrt{2}}{2}, 0\right)$ and $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ form an isoceless right triangle, so the segment connecting $(0,0)$ with $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ forms a 45^0 angle with the x'-alis. Similarly, the segment connecting $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ forms a -45^0 angles with the x'-axis. The sector of the circle bounded by these two segments has therefore a $45^0 + 45^0 = 90^0$ angle, so its area equals one fourth of the full area of the circle, that is, $\frac{\pi}{4}$. The area of the isoceless right tiragnle with vertices $(0,0)$, $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ is $\frac{1}{2}$, so the part of the circle which lies to the right of the line $x' = \frac{\sqrt{2}}{2}$ has area $\frac{\pi}{4} - \frac{1}{2}$. Finally, the transformation (3) had the effect of dividing the area of any figure by $\sqrt{3}$, so the area of the region we wanted to find in the problem is $\frac{\sqrt{3}\pi}{2} - \frac{\sqrt{3}}{2}$.

Problem 4. Since X is moved to itself by a rotation by 48^0 about O, it is also moved to itself after iterating such a rotation any number of times, that is, by a rotation by any integer multiple of 48^0, namely by 96^0, 144^0, 192^0, 240^0, 288^0, 336^0, 384^0 angles and so on. Since $384^0 = 24^0$, X is moved to itself by a 24^0 rotation, and hence also by a rotation by any itneger multiple of 24^0. Since $72^0 = 24^0 \times 3$, a 72^0 degree rotation about O preserves X.

What about a 90^0 rotation? Of course, there exist figures (such as a circle centered at O), which are preserved both by a 48^0 rotation and a 90^0 rotation. However, we claim that it *does not necessarily follow* from the assumptions of the problem that X is preserved by a 90^0 rotation. To show this, it is enough to give at least one example of a figure X which is preserved by a 48^0 rotation but not by a 90^0 one. Let I_0 be a segment of length 1, one of whose endpoints is O. For $0 \le k < 15$, let I_k be the segment obtained from I_0 by a rotation about O by the angle $k \cdot 24^0$. Let $X = \bigcup_{k=0}^{14} I_k$ be the union of $I_0, I_1, \dots, I_{14}$. Then X is moved to itself by a rotation by any integer multiple of 24^0, but by no other rotation. (Indeed, any rotation preserving X must move I_0 to one of $I_1, I_2, \dots, I_{14}$ and hence the angle of rotation must be divisible by 24^0.)

Problem 5. The idea is to rearrange the factors on the left hand side as follows:

$$(1 \cdot n)(2 \cdot (n-1)) \cdot \ldots \cdot ((n-1) \cdot 2)(n \cdot 1).$$

We obtain the product of n factors of the form $k(n+1-k)$, where $1 \le k \le n$. Thus it is sufficient to prove that each of the n factors is at least n, and if $n > 2$ then there are some factors strictly greater than n.

Now, the first and the last factors are equal to n. All the other factors correspond to the values of k from 2 to $n-1$, and if $n > 2$ then there is at least one integer k such that $2 \le k \le n-1$. It remains to show that $k(n+1-k) > n$ if $2 \le k \le n-1$.

Interchanging k with $n+1-k$, if necessary, we may assume that $k \le \frac{n+1}{2}$ and $n+1-k \ge \frac{n+1}{2}$. Then $k(n+1-k) \ge 2 \cdot \frac{n+1}{2} \ge n+1 > n$, as desired.

Problem 6. Zero lines divide the plane in one part, one line in 2 parts, 2 lines in 4 parts, 3 lines in 7 parts, and so on. From this one can notice that the number of parts has increased by 1 after introducing the first line, by 2 after introducing the second line, by 3 after introducing the third line, and so on. We can thus form a conjecture that n lines divide the plane into

$$1 + 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} + 1$$

parts. We prove this conjecture by induction. For $n = 0$ and $n = 1$ the result is clear.

Now, suppose the result is known for k lines. We want to show that introducing the $(k+1)$-st line has the effect of adding $k+1$ new parts. Let L denote the $(k+1)$-st line. By assumption, L meets the first k lines in exactly k distinct points. The line L is divided into $k+1$ segments by these k points (here a segment is allowed to be infinite in one direction). Each of these $k+1$ segments passes through one previously undivided region and divides it in 2. This has the effect of adding exactly $k+1$ new regions, as desired.

Problem 7. Finding the last three digits of a number n is the same as finding its residue mod 1000 (that is, the remainder after dividing n by 1000 with remainder). The answer does not change if we add 1000000^{1999} to the sum. Let us divide the sum

$$(4) \quad 1^{1999} + 2^{1999} + 3^{1999} + 4^{1999} + \cdots + 999998^{1999} + 999999^{1999} + 1000000^{1999}$$

at hand into one thousand parts, namely, the first thousand of summands, the second thousand of summands, and so on, until the thousandth thousand of summands. Next, let us compare the first thousand of summands with the second one. We observe that 1 is congruent 1001 mod 1000 (that is, they have the same last three digits), so 1^{1999} is congruent to 1001^{1999} mod 1000, 2^{1999} is congruent to 1002^{1999} mod 1000, and so on, all the way to 1000^{1999} congruent to 2000^{1999} mod 1000. Thus

$$1^{1999} + 2^{1999} + 3^{1999} + 4^{1999} + \cdots + 998^{1999} + 999^{1999} + 1000^{1999} \tag{5}$$

is congruent to

$$1001^{1999} + 1002^{1999} + 1003^{1999} + 1004^{1999} + \cdots + 1998^{1999} + 1999^{1999} + 2000^{1999} \tag{6}$$

mod 1000. Let r be the common residue of the sums (5) and (6) mod 1000 (r is nothing but the number formed by the last three digits of (5) (or (6)). By the same reasoning as above,

$$2001^{1999} + 2002^{1999} + 2003^{1999} + 2004^{1999} + \cdots + 2998^{1999} + 2999^{1999} + 3000^{1999} \tag{7}$$

is also congruent to r modulo 1000, and so is the sum of every other group of 1000 consecutive terms in (4). Thus modulo 1000 the sum (4) is congruent to a sum of 1000 terms, each of which is congruent to r. We obtain $1000 \cdot r$, which is congruent to 0 mod 1000. Thus the last three digits of the sum (4) are 0 (in fact, it can be shown that the last five digits of this sum are 0).

Problem 8. We have

$$8 = 5 + 3, \tag{8}$$

$$9 = 3 \cdot 3 \quad \text{and} \tag{9}$$

$$10 = 2 \cdot 5. \tag{10}$$

Thus 8, 9, and 10 emeralds can each be paid by three and five emerald notes. Now, let n be any number greater than 7. The number n can either be divisible by 3, or have residue 1 or 2 mod 3. Since 8, 9 and 10 are congruent to 2, 0 and 1 mod 3, exactly one of the numbers $n-8$, $n-9$ or $n-10$ is divisible by 3. Moreover, since $n \geq 8$, the number in the set $\{n-8, n-9, n-10\}$ which is divisible by 3 is non-negative, hence it can be written in the form $3k$, where k is a non-negative integer. Taking into account (8), (9) and (10), we see that n can be written in one of these forms:

$$\begin{aligned} n &= 5 + 3(k+1) \qquad \text{or,} \\ &= 3(k+3) \qquad \text{or} \\ &= 2 \cdot 5 + 3k. \end{aligned}$$

Thus n emeralds can be paid either by one 5 emerald and $(k+1)$ three emerald notes, or by $k+3$ three emerald notes, or by 2 five emerald notes and k three emerald notes. This completes the proof.

Problem 9. We give a proof by induction on n. For $n=0$ the result is obvious (also, for $n=1$, one line divides the plane into two parts, so we can color one of them white and the other red).

Now assume that the result is true for k lines; we will now prove it for $k+1$. Consider $k+1$ lines in the plane. By a **segment** we will mean a maximal interval I (possibly of infinite length), contained in one of the lines, such that there are no intersection points between our lines lying in the interior of I. Endpoints of such a segment (if they lie in the finite plane) are intersection points between our lines. Every segment belongs to the boundary of exactly two regions, which are bounded by our lines. We will call these regions the **parts adjacent to** I. Suppose we are given a coloring of the parts of the plane bounded by the lines. Let I be a segment. We will say that I is **good** with respect to the given colouring if the two regions adjacent to it are coloured in different colours; **bad** if they have the same colour. A colouring is called **good** if all the segments are good for it. We want to show that a good colouring always exists.

Pick one of the lines and call it L. By the induction assumption, there exists a good colouring G of the regions defined by the remaining k lines. Now, consider the segments of the given configuration of $k+1$ lines. Every segment not contained in L is also a segment for the set of k lines, and hence is good for G. On the other hand, every segment I contained in L cuts through one of the regions defined by the k lines. Hence both regions adjacent to I are coloured the same colour in G, so that I is bad for G. Now let G' be the following colouring. L divides the plane in two regions; label them A and B. For every part C of the plane bounded by our $k+1$ lines, contained in A, let C have the same colour as in G. For every part D of the plane bounded by our $k+1$ lines, contained in B, let D have different colour than the one it had in G; this defines the colouring G'. We claim that G' is the desired good covering. Indeed, consider any segment I of the configuration of $k+1$ lines. If $I \subset A \setminus L$ then I is good for G' because it was good for G. If $I \subset B \setminus L$, I is good for G' because it was good for G and the colours of both regions, adjacent to I, have changed from G to G'. Finally, suppose $I \subset L$. Then I was bad for G. One of the regions adjacent to I lies in A, the other in B. Then one of these regions has the same colour in G as in G', while the other changes colour. Thus since I was bad for G it becomes good for G'. This completes the proof that G' is good. We have constructed the desired colouring.

Problem 10. Suppose that x has $n+1$ digits, where $n \geq 1$. Write

$$x = a_n a_{n-1} \dots a_0 \tag{11}$$

in the decimal form, where $a_0, \dots, a_n$ are digits of x. Then $F(x) = a_{n-1}a_{n-2}\dots a_0$, where we allow the number $F(x)$ to start with zeroes.

(a) Suppose $x = 58 \cdot F(x)$. Then $x - F(x) = 57 \cdot F(x) = 3 \cdot 19 \cdot F(x)$. On the other hand, (11) says that $x - F(x) = a_n \cdot 10^n = a_n \cdot 2^n \cdot 5^n$. We obtain the equality

$$a_n \cdot 2^n \cdot 5^n = 3 \cdot 19 \cdot F(x). \tag{12}$$

Since a_n is an integer between 1 and 9, it cannot be divisible by 19. Thus the left hand side of (12) is not divisible by 19, while the right hand side is. Hence such a number x does not exist.

(b) Reasoning as above, we obtain the equality

$$a_n \cdot 2^n \cdot 5^n = 56 \cdot F(x) = 7 \cdot 2^3 \cdot F(x). \tag{13}$$

Since the right hand side of (13) is divisible by 7, so is the left hand side. Since 2 and 5 are prime to 7, we must have

$$7 \mid a_n. \tag{14}$$

Since a_n is an integer from 1 to 9, (14) implies that $a_n = 7$. Also, comparing the left hand side with the right hand side in (13), we obtain $n \geq 3$. Trying $n = 3$, we find a solution: $x = 7125$.

Note. Reasoning as above, one can prove the following result. Let k be a positive integer. There exists x such that $x = k \cdot F(x)$ if and only if k is of the form $k = 2^a 3^b 5^c 7^d$, where a, b, c, d are non-negative integers such that $b \leq 2$, $d \leq 1$ and $bd = 0$.