Problem 1. Prove that $1 + 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)!$ for all natural numbers n.

Problem 2. How many zeroes are there at the end of the number $11^{100} - 1$?

Problem 3. Let x_1, x_2, x_3, x_4 be integers (not necessarily positive). Suppose that the four smallest numbers in the set $\{x_1 + x_2, x_1 + x_3, x_1 + x_4, x_2 + x_3, x_2 + x_4, x_3 + x_4\}$ are 1, 5, 8 and 9. Find x_1, x_2, x_3, x_4.

Problem 4. Find all the triples p, q, r of prime numbers such that $pqr = 5(p + q + r)$ (here we allow some of the numbers p, q, r to coincide).

Problem 5. Let M denote the set of natural numbers of the form $x^2 + 5y^2$, where x and y are integers.

(a) Show that M is closed under multiplication, i.e. that the product of two numbers in M is again in M.

(b) A number n from M is called **basic** if $n > 1$ and n is not divisible by any number $m \in M \setminus \{1, n\}$. Do there exist numbers in M which can be expressed as a product of basic numbers in two different ways?

(c) Show that there are infinitely many basic numbers.

Problem 6. Let $\{x_n\}$ be an infinite sequence of real numbers. Show that $\{x_n\}$ contains an infinite monotonous subsequence.

Problem 7. (a) What is the greatest number of bishops which can be placed on the 8×8 chessboard in such a way that no bishop can take another (that is, no two bishops should be on the same diagonal)?

(b) Let x be the answer you obtained in (a). Prove that the number of ways in which x bishops can be placed on the board so that no bishop can take another is a complete square.

In the following three problems, $f(x)$ is a polynomial with integer coefficients.

Problem 8. Suppose $f(0)$ and $f(1)$ are odd numbers. Prove that f has no integer roots.

Problem 9. Assume $|f(3)| = |f(7)| = 1$. Show that f has no integer roots.

Problem 10. Assume that deg $f(x) = 7$ and that there are five distinct integers x_1, x_2, x_3, x_4, x_5 such that

$$|f(x_1)| = |f(x_2)| = |f(x_3)| = |f(x_4)| = |f(x_5)| = 1.$$

Show that f has no integer roots.

Problem 11. Determine all the three digit numbers N having the property that N is divisible by 11 and $\frac{N}{11}$ is equal to the sum of the squares of the digits of N.