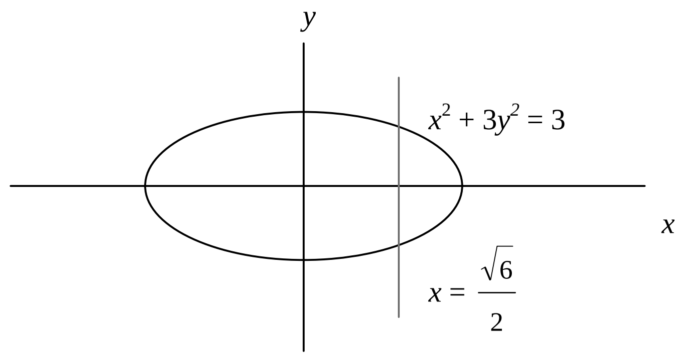
A Mathematical 'Hodgepodge'

A collection of problems to be discussed at the November 1999 Teachers' SIMMER meeting

Problem 1. Which is greater: $99^{50} + 100^{50}$ or 101^{50} ?

Problem 2. (a) In how many ways can 8 rooks be placed on an 8 x 8 chessboard in such a way that no rook can take another?
(b) In how many ways can 4 rooks be placed on an 8 x 8 chessboard in such a way that no rook can take another?

Problem 3. Find the area of the region in the plane, bounded on the right by the ellipse $x^2 + 3y^2 = 3$ and on the left by the straight line $x = \frac{\sqrt{6}}{2}$



Hint: this problem can be solved without using integration.

Problem 4. Let X be a figure in the plane. Assume that X is moved to itself by a rotation about a point O by 48^o. Does it necessarily follow that X is moved to itself by a rotation about O by 90^o? by 72^o ?

Problem 5. Prove that for every integer $n > 2$,
$(1.2.\ldots.n)^2 > n^n$.

Problem 6. Into how many parts is a plane divided by n straight lines, such that no two lines are parallel and no three lines pass through the same point?

Problem 7. Find the last three digits of the sum
$1^{1999} + 2^{1999} + 3^{1999} + 4^{1999} + \cdots + 999998^{1999} + 999999^{1999}$

Problem 8. The monetary unit in the Republic of Oz is called an emerald, and both three emerald and five emerald bills are in circulation. Prove that any sum greater than 7 emeralds can be paid by three and five emerald notes.

Problem 9. Suppose that a plane is divided into parts by n straight lines. Prove that these parts can be coloured with red and white paint, such that any two parts, having a common side, are coloured with different colours.

Problem 10. For a positive integer x with at least two digits, let $F(x)$ denote the integer obtained from x by deleting the first digit. Does there exist x such that

(a) $x = 58.F(x)$?
(b) $x = 57.F(x)$?