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SIMMER

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Introduction to Combinatorial Block Designs

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We will introduce this topic by looking at three classic problems in recreational mathematics. Then we will delve in to the underlying concepts and look at the answers to some conceptual questions. Finally, we will consider how these ideas fit together in the solution to a fourth and more difficult problem.

Three Problems

I. The Farmer's Wheat. A farmer wants to plant seven different varieties of wheat. He has many fields available, and can plant more than one variety in each (but wants the same number in each). *For every pair of varieties, there needs to be exactly one field in which they're both planted* (so he can compare their relative yields under identical soil conditions, without wasting more than one field per comparison). He can't plant all seven in one field; it's too many for a single field, and doesn't give him a chance to see each variety growing in several different soil conditions. How many fields must he use?

One solution is to plant two varieties in each field, using 21 fields in total (there are 21 different pairs of varieties, each needing a field). This solution is tabulated below:

Field #	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
First variety	1	1	1	1	1	1	2	2	2	2	2	3	3	3	3	4	4	4	5	5	6
Second variety	2	3	4	5	6	7	3	4	5	6	7	4	5	6	7	5	6	7	6	7	7

However, using that many fields would be expensive!

There is one solution that uses fewer fields—in fact, only one. It is called the *Fano Plane*, because it has an interesting interpretation as a kind of geometry. Can you find it?

II. The Schoolgirls' Walk. In an all-girls boarding school, there are 9 girls. On four days during the week they go out for a walk in three lines of three girls each. Find a way to schedule their four walks in such a way that each girl has every other girl as a line-mate exactly once.

[1] Notes edited and modified for **SIMMER** by Philip Spencer.

III. The Tournament Schedule. In a certain sports tournament, each team must play every other team exactly once. No team can play twice in the same day. This means each team must play on $n-1$ different days (where n is the number of teams)—one day for each opponent. The ideal tournament would be if every team plays every day; then the whole tournament takes only $n-1$ days (otherwise it would take longer).

Can this be done? Can you find explicit schedules for the $n = 4$, $n = 6$, and $n = 8$ cases?

Delving Into The Concepts

Combinatorial Designs. These three problems are all special cases of the following mathematical concept:

> A *combinatorial design* consists of **a set** (whose elements are called *points*), together with a collection of **subsets** (called *blocks*), such that each block contains the same number of points, and, for each pair of points, there is exactly one block containing both of them.

A combinatorial design has four important numbers attached to it: v, the number of points; b, the number of blocks; k, the size of each block; and r, the number of blocks each point occurs in (this happens to be the same number for each point).

Question 1. Why must it be true that each point occurs in the same number of blocks?

Any solution to the Farmer's Wheat Problem is a combinatorial design: the points are the varieties of wheat and the blocks are the fields. All solutions have $v = 7$ (seven varieties of wheat). The 1-field solution the farmer rejected has $b = 1$, $k = 7$, and $r = 1$ (1 field, 7 varieties in each, and each variety occurs in 1 field). The overly expensive 21-field solution has $b = 21$, $k = 2$, and $r = 6$ (21 fields, 2 varieties in each, and each variety occurs in 6 fields). It is up to you to find the values of b, k, and r for the Fano plane solution.

Why are there no other solutions? One reason is that not all combinations of numbers (v, b, k, r) are possible in a combinatorial design.

Question 2. Find two conditions that the numbers v, b, k and r have to satisfy, and infer from them that b and r can be completely determined from v and k. Also show that because of this there can't be any other solution to the Farmer's Wheat Problem.

Combinatorial Designs Are Geometries. In plane geometry, every two points determine a line. In a combinatorial design, every two points determine a block (there's exactly one block containing both of them). In fact, the points and blocks of a combinatorial design satisfy many of the same axioms that the points and lines in a plane do. That's why a combinatorial design is really a kind of abstract geometry (but a very different sort from Euclidean geometry, having only finitely many points instead of infinitely many!). Hence the geometric name "the Fano plane" for the solution to the Farmer's Wheat Problem.

In Euclidean plane geometry there is the important concept of *parallel lines.* This has no analogous counterpart in the Fano plane, but it does in the combinatorial designs associated to our other two examples.

Resolvable Combinatorial Designs. The Schoolgirls' Walk Problem is a combinatorial design in which the points are the girls and the blocks are the lines. There are twelve lines in total (three on each of the four days). However, there's an extra feature in this problem: the lines are *grouped* into three lines per day, and every girl has to be in exactly one of these three lines.

A similar feature is present in the Tournament Scheduling Problem. It is a combinatorial design in which the points are the teams and the blocks are the games. The blocks are grouped into the first day's games, the second day's games, etc. We are asking for each team to play in exactly one of each day's games.

These are each examples of the following concept:

> A *resolvable combinatorial design* is a combinatorial design in which the blocks can be arranged into groups in such a way that each point occurs exactly once in each group.

Resolvable combinatorial designs are geometries that do have a counterpart to the "parallel line" concept of Euclidean geometry: blocks in the same group can be thought of as "parallel". The different groups can be thought of as different "slopes". Given any point p and any group, there is exactly one block in that group which contains p. This corresponds to the property in Euclidean geometry that, given any point p and slope m, there is exactly one line of slope m which contains p.

Not all combinatorial designs are resolvable. For one thing, there's an extra condition that v and k have to satisfy in a resolvable combinatorial design.

Question 3. Find the extra condition that v and k have to satisfy in a resolvable combinatorial design.

Neither the Fano plane nor the 21-field solution to the Farmer's Wheat Problem satisfies this condition, so neither is resolvable.

Even when v, b, k, and r satisfy all the necessary conditions, there's no guarantee that that there exists a resolvable combinatorial design (or any combinatorial design at all) which has those values.

Question 4. Referring back to the Schoolgirls' Walk Problem, suppose that three more girls come to the school. The necessary condition of question 3 is still satisfied. Can we now find a similar schedule for a series of walks? Why or why not? (Note that the number of walks need no longer be four, but we still require that on each walk the girls go out in lines of three, and that every girl has every other girl exactly once as a linemate.)

Question 5. What if 4 more girls come instead of 3?

Question 6. More generally, what is the smallest number of new girls that would have to come to the school in order for a similar schedule of walks to be possible in the sense of satisfying all the necessary conditions? Does such a schedule actually exist, and can you find it?

We close with an example of a problem in which all the necessary conditions are satisfied, and it turns out that a resolvable combinatorial design does exist, but finding it is a challenge.

Kirkman's Schoolgirl Problem

This was posed by the Rev. Kirkman in 1850:

> A school mistress has fifteen girl pupils and she wishes to take them on a daily walk. The girls are to walk in five rows of three girls each. It is required that no two girls should walk in the same row more than once per week. Can this be done?

This problem is actually the same as the second part of question 6 above. Stated abstractly, the problem is to find a resolvable combinatorial design in which $v = 15$, $b = 35$, $k = 3$, and $r = 7$. The construction of this design involves an interesting mixture of the patterns in the Fano Plane and the patterns in the $n = 8$ solution to the Tournament Scheduling Problem. We outline below how the design is constructed.

Divide the fifteen points of the desired resolvable combinatorial design up into a group of seven (1,2,3,4,5,6,7) and a group of eight (8,9,10,11,12,13,14,15).

On the group of seven construct a Fano Plane. By this I mean take the blocks of the Fano Plane that you found earlier, except now the numbers refer to girls instead of wheat varieties.

Next, construct a tournament schedule for eight teams named 8, 9, 10, 11, 12, 13, 14, and 15. Take the 28 different games played to be additional blocks, except now the numbers will refer to girls instead of sports teams.

Now what we have is seven blocks of size 3 and twenty-eight blocks of size 2. This is a total of 35 blocks, which is exactly how many we need (from the necessary conditions), but we need them all to be of size three. The solution is this. To each block of size two, we add the girl whose number is the same number as the day that game was played. This gives us 35 blocks of size three.

Question 7. Show that every pair appears together in the same block exactly once, proving that we have a combinatorial design.

Finally, we need to show that this combinatorial design is resolvable.

Question 8. Find a way to allocate these 35 blocks of girls into seven groups (days) of five blocks each, such that every girl appears exactly once in each group. Use this to write down a schedule which solves Kirkman's Schoolgirl Problem.

For further reading on Combinatorial Designs try the following books.

References

[1] Peter J. Cameron. *Combinatorics: Topics, Techniques, Algorithms.* Cambridge University Press, Cambridge, 1994.

[2] Anne Penfold Street and Deborah J. Street. *Combinatorics of Experimental Design.* Oxford University Press, Oxford, 1987.

[3] J. H. van Lint and R. M. Wilson. *A Course in Combinatorics.* Cambridge University Press, Cambridge, 1992.

[4] W. D. Wallis. *Combinatorial Designs*, volume 118 of *Pure and Applied Mathematics.* Marcel Dekker, Inc., New York, 1988.

Hints

Hint for the Farmer's Wheat Problem. What if, instead of planting two varieties per field, you plant three varieties per field?

Use the fact that every pair appears once and only once on some field, count the total number of pairs and the number of pairs that appear on each field, and then by dividing, find the total number of fields.

Similarly, find the number of fields in which each variety appears by taking the total number of pairs involving that variety and dividing by the number of those pairs that occur in a single field.

Knowing these two numbers should help you figure out the schedule.

Hint for the Schoolgirl's Walk Problem. Try the same strategy as in the hint for the Farmer's Wheat Problem, above.

Hint for the Tournament Scheduling Problem. Calculate the number of games that must be played on each day. Use this to show that such a schedule cannot be found when n is odd.

It can be done when n is even. The $n = 2$ case is easy: with two teams, there is just one game, taking place on one day. Now try to find schedules that work for the $n = 4$, $n = 6$, and $n = 8$ cases. Can you arrange your answers to have a general pattern which extends and works for all even n?

Hint for Question 1. Suppose a point P occurs in r blocks. Find a formula for r in terms of v and k, thereby proving that the number r does not depend on which point P you were considering.

To find this formula, see the last part of the hint for the Farmer's Wheat Problem.

Hint for Question 2. Follow the same strategy as in the hint for the Farmer's Wheat Problem.

Hint for Question 3. If we can find a group of blocks (each block of size k) that represents each of the v points exactly once, then what do we know about v/k?

Hint for Question 4. It is not possible.

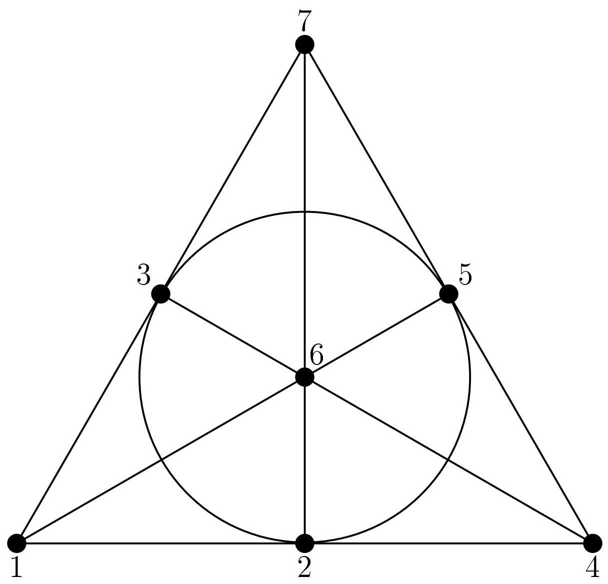
Hint for Question 5. Again, it is not possible, but a non-resolvable combinatorial design does exist. Can you find it?

Hint for Question 7. Each pair of girls corresponds to a pair of numbers from 1 to 15. Consider the three possible cases: both numbers are 7 or below, both are 8 or above, or one is 7 or below and the other is 8 and above.

Hint for Question 8. Exploit the vertical structure from the Fano plane solution, and the horizontal structure from the $n = 8$ Tournament Scheduling solution, to show that if you find a set of five lines of girls which works for day 1, then cycle the girls by $1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 1$ and $8 \to 9 \to 10 \to 11 \to 12 \to 13 \to 14 \to 8$ (with 15 left unchanged), you get a set of lines which works for day 2, and by cycling again you get a set that works for day 3, and so on.

Solutions

Solution to the Farmer's Wheat Problem. The following diagram gives a succinct answer to this problem:



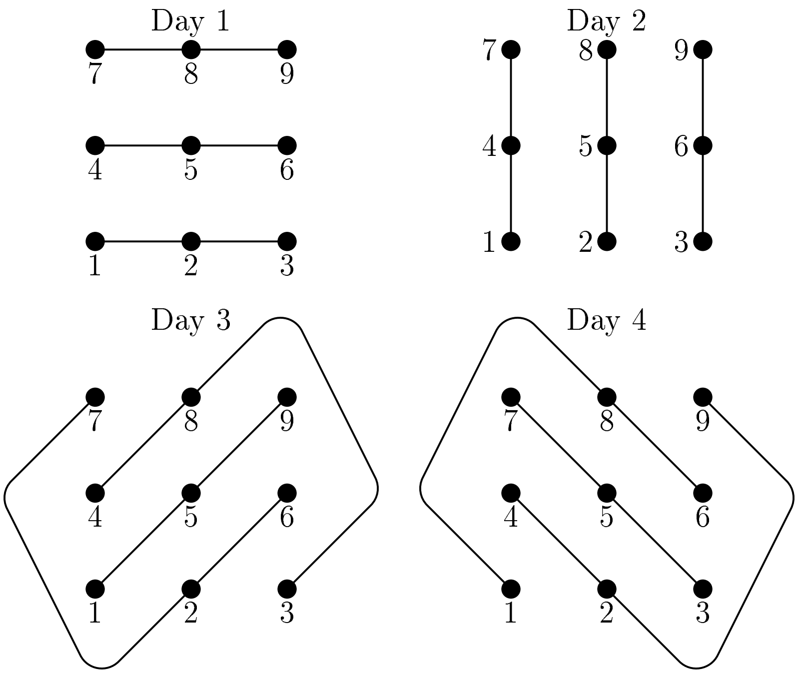
Here the points represent the varieties; the six line segments and one circle represent the fields. This solution can be interpreted as a geometry (the "lines" of this geometry are the six line segments and one circle). In this geometric interpretation, the "points" stand for varieties of wheat and the "lines" stand for fields. There is an interesting duality to this geometry: you could equally well interpret the "lines" as being the varieties of wheat and the "points" as being the fields, and the same properties are preserved.

We can also list the fields and the varieties planted in each one:

field #	first variety	second variety	third variety
1	1	2	4
2	2	3	5
3	3	4	6
4	4	5	7
5	5	6	1
6	6	7	2
7	7	1	3

Note the pattern as you go down the table!

Solution to the Schoolgirls' Walk Problem. The following diagram pictorially represents the answer:



The blocks, divided up into the four days of walks, are:

	day 1			day 2			day 3			day 4		
line 1	1	2	3	1	4	7	1	5	9	1	6	8
line 2	4	5	6	2	5	8	2	6	7	2	4	9
line 3	7	8	9	3	6	9	3	4	8	3	5	7

Solution to the Tournament Scheduling Problem. If each of the n teams plays every day, then there must be $n/2$ games each day (as each game involves two teams). In particular, $n/2$ must be an integer, so no solution is possible if n is odd.

If $n = 2$ the solution is obvious: there is just one game, taking place on one day. If $n = 4$ the following schedule works:

	day 1	day 2	day 3
game 1	A vs. B	A vs. C	A vs. D
game 2	C vs. D	B vs. D	B vs. C

For $n = 6$ the following schedule works:

	day 1	day 2	day 3	day 4	day 5
game 1	A vs. B	A vs. C	A vs. D	A vs. E	A vs. F
game 2	C vs. D	B vs. E	B vs. F	B vs. D	B vs. C
game 3	E vs. F	D vs. F	C vs. E	C vs. F	D vs. E

Many other schedules are possible as well. No pattern is immediately obvious from the particular schedules we've chosen above. For the $n = 8$ case, let's be more careful to choose a schedule in a manner which exemplifies a clear pattern:

	day 1	day 2	day 3	day 4	day 5	day 6	day 7
game 1	A vs. H	B vs. H	C vs. H	D vs. H	E vs. H	F vs. H	G vs. H
game 2	B vs. G	C vs. A	D vs. B	E vs. C	F vs. D	G vs. E	A vs. F
game 3	C vs. F	D vs. G	E vs. A	F vs. B	G vs. C	A vs. D	B vs. E
game 4	D vs. E	E vs. F	F vs. G	G vs. A	A vs. B	B vs. C	C vs. D

Notice the patterns going across the table.

To get a general solution for any even number $n = 2k$ of teams: on day one, have team 1 playing team $2k$, team 2 playing team $2k - 1$, team 3 playing team $2k - 2$, and so on, up through team k playing team $k + 1$. For day two, rotate the teams' positions in the schedule, according to the following rotation: $1 \to 2 \to 3 \to 4 \to \ldots \to 2k - 1 \to 1$, with team $2k$ staying put. (For example, in the above schedule we replaced A with B, B with C, C with D, D with E, E with F, F with G, and G with A, with H staying put, to get the schedule for day two from the schedule for day one). Similarly, perform another rotation to get the third day's schedule, and so on. It is left to you to verify that this gives a valid schedule in which each team plays every other team exactly once.

Answer to Question 1. Pick a point P. Since there are $v - 1$ other points besides P, there are $v - 1$ pairs which involve P. None of these pairs can occur in more than one block. A block that contains P has $k - 1$ of these pairs in it (since that block contains $k - 1$ points besides P). Therefore, if r is the number of blocks in which P occurs, we must have $r(k-1) = v-1$, so $r = (v-1)/(k-1)$, proving that r is independent of the point P.

Answer to Question 2. The answer to qustion 1 reveals that one condition is that $r = (v-1)/(k-1)$, which also implies that $(v-1)/(k-1)$ has to be an integer.

The other condition is as follows. There are $v(v-1)/2$ pairs in total. Each block has $k(k-1)/2$ pairs in it. Since each pair occurs once and only once we get that

$$b = \frac{\text{total number of pairs}}{\text{pairs in each block}} = \frac{v(v-1)/2}{k(k-1)/2} = \frac{v(v-1)}{k(k-1)}.$$

In particular, it is necessary that $v(v-1)/k(k-1)$ be an integer.

In the Farmer's Wheat Problem, $v = 7$. Since $(v-1)/(k-1)$ must be an integer, that means $k-1$ must be a factor of 6. Since the only factors of 6 are 1, 2, 3, and 6, this means k must be 2, 3, 4, or 7. The case $k = 4$ is impossible since then $v(v-1)/k(k-1)$ would be 42/12 which is not an integer. This leaves only the cases $k = 2$ (the 21-field solution), $k = 3$ (the Fano plane solution), or $k = 7$ (the plant-everything-in-one-field solution which the farmer rejected).

Answer to Question 3. Since we can find a group of blocks (which have k points each) that together represents every one of the v points exactly once, k must divide v. That means v/k must be an integer (it is the number of blocks per group).

Answer to Question 4. Although $v/k = 12/3 = 4$ is an integer and $v(v-1)/k(k-1) = 132/6 = 22$ is an integer, $(v-1)/(k-1) = 11/2$ is not an integer, so this situation fails one of the necessary conditions. It is therefore not possible to find such a schedule of walks.

Answer to Question 5. $v(v-1)/k(k-1) = 156/6 = 26$ and $(v-1)/(k-1) = 6$ are both integers so the necessary conditions for a combinatorial design are met. $v/k = 13/3$ is not an integer so it cannot be resolvable. A non-resolvable combinatorial design with these parameters has the following blocks. It is divided up into sets for convenience and also hoping that you will notice a pattern (that also occurred in the Fano plane).

set 1			set 2		
1	3	9	2	5	6
2	4	10	3	6	7
3	5	11	4	7	8
4	6	12	5	8	9
5	7	13	6	9	10
6	8	1	7	10	11
7	9	2	8	11	12
8	10	3	9	12	13
9	11	4	10	13	1
10	12	5	11	1	2
11	13	6	12	2	3
12	1	7	13	3	4
13	2	8	1	4	5

Answer to Question 6. We know that k has to be 3 and v/k has to be an integer, so v must be a multiple of 3. Also $(v-1)/(k-1)$ must be an integer, so $v-1$ must be a multiple of 2, so v must be odd. Finally, $v(v-1)/k(k-1)$ must be an integer, so $v(v-1)$ must be a multiple of 6; this condition will automatically be satisfied if v is an odd multiple of 3.

Therefore, the next value of of v for which a schedule of walks might be possible is the next odd multiple of three, namely 15 (which corresponds to six more students coming to the school).

Answer to Question 7. Recall that our blocks are of two types, which we can call *Fano blocks* (blocks from the Fano plane, consisting of numbers from 1 to 7) and *mixed blocks* (consisting of a tournament game, which is two numbers from 8 to 15, together with one number from 1 to 7 representing the day on which that game was played).

Any pair of points in which both numbers are 7 or below occurs exactly once among the Fano plane blocks, and never in the mixed blocks. Any pair of points in which both numbers are 8 or above occurs exactly once in the tournament schedule, and hence exactly once in the mixed blocks, and never in the Fano plane blocks. Any pair of points in which one number a is 7 or below and the other number b is 8 or above occurs exactly once in the mixed blocks (because team b plays exactly once on date a), and never in the Fano plane blocks.

Therefore, each pair of points occurs exactly once amongst all the blocks.

Answer to Question 8. Note the vertical structure to the Fano plane. The first block (the first row of the table in the answer to the Farmer's Wheat Problem) consists of the numbers 1, 2, and 4. If we add 1 to each number we get the second row. If we add 1 again we get the third row. If we continue to add 1 each time (except that when we are at the number 7 we go back down to 1 instead of up to 8) we eventually generate all the blocks of the Fano plane.

Similarly, if we take the games from one day of the sports tournament and cycle the teams according to the rotation

$$8 \to 9 \to 10 \to 11 \to 12 \to 13 \to 14 \to 8$$

(with team 15 remaining fixed), we get the next day's games.

That means that if we take one of our "mixed" blocks of girls (where two of the numbers, call them a and b, are from 8 to 15, representing one of the sports games, and the third number is c is from 1 to 7 representing the day on which the a versus b game was played), and cycle the numbers according to

$$1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 1$$

$$8 \to 9 \to 10 \to 11 \to 12 \to 13 \to 14 \to 8$$

(with 15 remaining fixed), a and b transform into a pair of numbers representing a game on day $c+1$, so the transformed triple is also one of our mixed blocks.

Thus, if we start with any one of our 35 blocks (either one from the Fano plane, or one of the mixed blocks), and keep applying the above rotation, we get a cycle of seven blocks.

All we need to do to solve Kirkman's schoolgirl problem is to find a schedule of five blocks which works for day 1, making sure none of these five blocks are in the same cycle. Then, to get the schedule for the second day, simply apply the above rotation to the first day's schedule, and so on for each of the seven days.

The final solution is shown below.

	day 1			day 2			day 3			day 4			day 5			day 6			day 7		
line 1	1	2	4	2	3	5	3	4	6	4	5	7	5	6	1	6	7	2	7	1	3
line 2	3	8	12	4	9	13	5	10	14	6	11	8	7	12	9	1	13	10	2	14	11
line 3	5	11	13	6	12	14	7	13	8	1	14	9	2	8	10	3	9	11	4	10	12
line 4	6	9	10	7	10	11	1	11	12	2	12	13	3	13	14	4	14	8	5	8	9
line 5	7	14	15	1	8	15	2	9	15	3	10	15	4	11	15	5	12	15	6	13	15