May 1999 Presentation Topic
Pricing Financial Derivatives
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This is a summary of what was presented and discussed at the May 27th **SIMMER** meeting, along with some problems and questions to think about.

Introduction

Derivative trading has been in the news a lot lately. Mostly we hear stories about incredible sums of money lost by Long Term Capital, Barings, and Orange County. Spectacular as these disasters are, they really involve miniscule fractions of the remarkable amount of trading that is going on in these new financial instruments.

But what are all these fancy new financial objects, and why are the banks and investment companies hiring hordes of fresh PhD's in math and physics (the so called 'rocket scientists')? What are all these analysts doing?

To try to get some idea let's look at a couple of very simple examples. Simple as they are, these models are actually pretty close to the real models that are being used on Bay Street and Wall Street.

Arbitrage Pricing Theory

Suppose we have a very simple stock. You can buy some shares at price S_0 per share at time 0, and one year later the stock price has either gone up to $S_1(u) = uS_0$ or down to $S_1(d) = dS_0$ where u and d are some positive numbers, and obviously $0 < d < u$. Furthermore let's say we know the respective probabilities: It goes up with probability p and down with probability $1 - p$.

You also have the option to put your money in the bank and earn interest. This involves no risk at all and \$1 invested becomes $\$(1 + r)$ at the end of the year. It is natural to assume that $d < 1 + r < u$ because if $1 + r > u$ there is no point in buying the stock, because you can do better with the bank for sure, and if $1 + r < d$ then there is no point putting your money in the bank, because you can do better with the stock for sure.

Now a guy in a sharkskin suit makes you an offer. Instead of the stock, he will sell you a call option on the stock with strike price K. That means that if after 1 year the stock is worth $S_1 > K$, you can cash it in for $S_1 - K$. Otherwise it is not worth anything. In other words in one year the call option will be worth

$$V_1 = \max(S_1 - K, 0).$$

(Note that K should be some number between dS_0 and uS_0 or this is a bit silly.) Now he says look, the probability that the stock will be worth $S_1(u) = uS_0$ is p, in which case the payoff is $V_1(u) = uS_0 - K$, and the probability that stock will be worth $S_1(d) = dS_0$ is $1 - p$ in which case the payoff is $V_1(d) = 0$. So the expected payoff is

$$pV_1(u) + (1 - p)V_1(d).$$

Of course a dollar a year from now is really worth $1/(1 + r)$ dollars now, so the price now should be

$$\frac{1}{1+r}[pV_1(u) + (1 - p)V_1(d)].$$

He offers to sell you the option for one half of that.

It looks like a great deal. Should you take it?

Suppose your initial wealth is \$$V_0$ and you are wondering whether to go for it. You actually have a completely different choice. You could buy Δ_0 shares of the stock and put the rest, $(V_0 - \Delta_0 S_0)$ in the bank. A year later you would have

$$\Delta_0 uS_0 + (1 + r)(V_0 - \Delta_0 S_0)$$

if the stock went up, and

$$\Delta_0 dS_0 + (1 + r)(V_0 - \Delta_0 S_0)$$

if the stock went down.

On the other hand if you put the V_0 into call options it is worth $V_1(u)$ if the stock goes up, and $V_1(d)$ if the stock goes down.

Let's try something. We have been given S_0, u, d and r and we can choose Δ_0 and V_0. So let's match things after one year.

$$\begin{aligned}
\Delta_0 uS_0 + (1 + r)(V_0 - \Delta_0 S_0) &= V_1(u) \\
\Delta_0 dS_0 + (1 + r)(V_0 - \Delta_0 S_0) &= V_1(d)
\end{aligned}$$

This is two equations in two unknowns, so we can solve it and the answer is

$$\Delta_0 = \frac{V_1(u) - V_1(d)}{S_1(u) - S_1(d)}$$

and

$$V_0 = \frac{1}{1+r}[qV_1(u) + (1 - q)V_1(d)],$$

where $q = (1 + r - d)/(u - d)$.

If the price of the call option was more than V_0, why would you bother? You would do better for sure buying Δ_0 shares and putting the difference in the bank.

On the other hand if the price of the call option was less than V_0 then who would be so stupid as to sell them when they could do better for sure borrowing money and investing in the stock.

So the fair price for the option has to be V_0. You say it is the *arbitrage price* because if the price were any different many people in sharkskin suits (known as arbitrageurs) would have a field day making money on the difference.

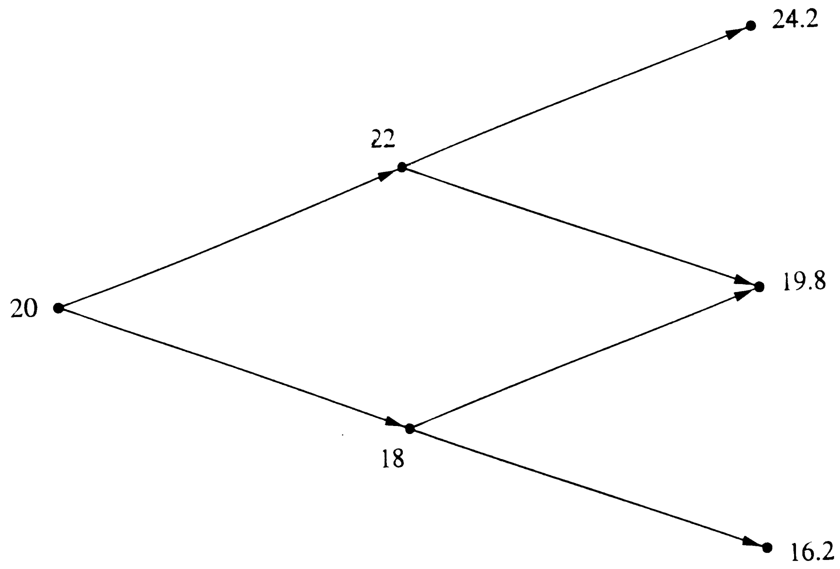
But now we have two different answers for the price of the option, the expectation price and the arbitrage price. And the arbitrage price doesn't even depend on the probabilities p and $1 - p$, but uses some new 'effective' probabilities q and $1 - q$ which only depend on the interest rate and possible price changes. These are called the *risk neutral probabilities.*

The market forces the arbitrage price to be the true price, so the strange conclusion is that if p is large enough, then the guys offer of half the expectation price is a bad deal!

Problems

Question 1. A stock price is currently \$20 and it is known that at the end of three months it will be either \$22 or \$18. The interest rate is 12% per annum. What is the value today of an option to buy the stock after three months for \$21?

Question 2. A stock price follows the following tree.



It starts at \$20 and after three months either goes up to \$22 or down to \$18. If it went to \$22 then three months later it either goes up to \$24.2 or down to \$19.8. If it went down to \$18 then three months later it either goes up to \$19.8 or down to \$16.2. The interest rate is still 12% per year. What is the value of an option to buy the stock for \$21 in six months?

Question 3. A stock price is currently \$20 and it is known that at the end of three months it will be either \$22 or \$20 or \$18. The interest rate is 12% per annum. Can you determine the value today of an option to buy the stock after three months for \$21?

Continuous time models

In the discrete time models the stock price just changed at discrete time. Now it is going to change at all times. In other words we have a function $S(t)$ which is the stock price at time t. It is going to evolve in some kind of random way but the truth is that no one really knows how, so they try some models.

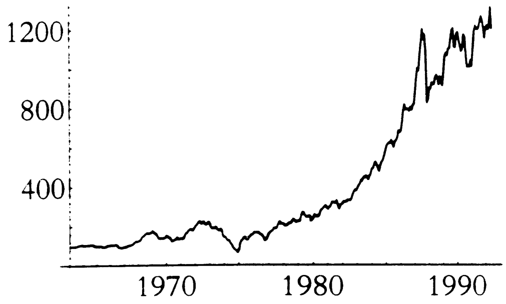
The simplest model which is reasonable is called geometric or exponential Brownian motion. Let's try to describe it. The main idea here is that the market is made up of a lot of very little investors, so price movements that we observe are sums of a great number of random small movements. From probability we know that a random quantity which is the sum of a lot of small random quantities always has a Gaussian (or Normal) disribution. But also the stock price grows at some rate μ. All prices also have to be relative to the total price. So in summary we expect

$$S(t+h) - S(t) \simeq \mu S(t)h + \sigma S(t)(B(t+h) - B(t))$$

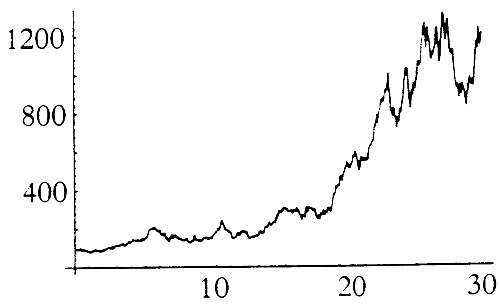
where $B(t+h) - B(t)$ are mean 0 normal random variables. Since the variance of a sum of independent random variables is the sum of the variances, we have to have that the variance of $B(t+h) - B(t)$ is linear in h. σ plays the role of the constant and it is called the *volatility*. So $B(t)$ itself is a random function, called Brownian motion and it is such that the increments $B(t+h) - B(t)$ are normal with mean zero and variance one, and any two such increments which are not overlapping are independent. If you take the limit of the equation you get

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t)$$

called a *stochastic differential equation*.



UK FTA index, 1963–92



Exponential Brownian motion

Not bad, eh?

Stochastic differential equations are different from ordinary differential equations because the function $B(t)$ turns out to be non-differentiable at every single point with probability one. So the equation is really strange and it took until the 1950's before people even started to make sense of it. In fact these functions are so strange that the ordinary rules of calculus, like the chain rule and the product rule and integration by parts are just not true. But there are analogues. For example the chain rule for Brownian motion turns out to be

$$df(B(t)) = f'(B(t))dB(t) + (1/2)f''(B(t))dt.$$

In normal calculus the last term wouldn't be there. This is called Ito's formula.

You can price a call option with strike price K at time T by analogy with the discrete time case. There is risk neutral probability distribution under which the arbitrage price of the option is

$$e^{-rT} \times \text{ Expected value of } \max(S(T) - K, 0).$$

Using Ito's formula you can check that this can be determined by solving a certain partial differential equation. In this lucky case, one can even write down the solution to this equation. Black, Scholes and Merton did this in the 1970's, and the fact that they finally managed to compute the fair price for an option meant that the whole business could finally get rolling. It's been growing like crazy as more and more fancier types of derivatives are invented, and Merton and Scholes went on to win the Nobel prize in Economics for their work.

Question 4. How would you go about constructing a continuous function which was nowhere differentiable? (Hint: Think fractals.)