October 1999 Presentation Topic
Pell's Equation
Ed Barbeau

Department of Mathematics
University of Toronto

This is a summary of what was presented and discussed at the October 1999 **SIMMER** meeting, along with some problems and questions to think about.

Any number of the form $1 + 2 + \cdots + n = n(n+1)/2$ is *triangular*. Which triangular numbers, besides 1, are also squares? A triple (a, b, c) is *pythagorean* iff $c^2 = a^2 + b^2$. For which pythagorean triples do the smallest two numbers differ by 1?

Investigation of these, and other similar, problems lead to a quadratic algebraic equation in two variables that can be reduced to the standard form

$$x^2 - dy^2 = k \ ,$$

where d and k are given integers, and a solution (x, y) is sought in integers. Such equations, called *Pell's equations*, have a long history. Mathematicians in India have studied them since the sixth century of the Christian era, and European's have been looking at them since the time of Fermat. At first, they were solved in an *ad hoc* way, but gradually a systematic theory was developed and Pell's equations are now recognized as an elementary form of an important type of diophantine equation which has a deep and rich theory.

It turns out that, when d is a nonsquare positive integer, the equation $x^2 - dy^2 = 1$ always has a solution in positive integers. Such solutions are often very easy to find, as when $d = 2, 3$. Sometimes, finding them is more troublesome, such as when $d = 13$. And occasionally, finding a solutions seems downright impossible, such as when $d = 61$. However, Brahmagupta solved $x^2 - 61y^2 = 1$ for positive integers back in the eleventh century, while Euler (1707–1783) was the first European to find a solution.

In preparation for this **SIMMER** session, try your hand at solving $x^2 - dy^2 = 1$ for various values of d from 2 up to 100. In some cases, you will see some patterns based on the form of the number, but in others you will need a certain amount of ingenuity. Why don't you get some of your students and colleagues working on it?

Other equations of particular interest (when we meet, you will see why) are $x^2 - dy^2 = 4$ where x and y are both required to be odd, and $x^2 - dy^2 = -1$. A very deep question for each of these is to characterize those values of d for which there is a solution in positive integers.

I have prepared a set of exercises on Pell's equation which can be used by students seeking enrichment or by teachers wanting to get into an interesting topic that still has connection with high school work. Chapters 1 (on the square root of 2) and 7 (on a cubic version of Pell's equation) are available on the net for your perusal:

- Chapter 1, PostScript Format:
  http://www.math.toronto.edu/mathnet/simmer/oct99.ch1.ps
- Chapter 1, PDF Format:
  http://www.math.toronto.edu/mathnet/simmer/oct99.ch1.pdf
- Chapter 7, PostScript Format:
  http://www.math.toronto.edu/mathnet/simmer/oct99.ch7.ps
- Chapter 7, PDF Format:
  http://www.math.toronto.edu/mathnet/simmer/oct99.ch7.pdf

PostScript files require either ghostscript or the capability to send the file directly to a PostScript printer. PDF files require Adobe Acrobat Reader which is available from Adobe's web site.