University of Toronto
Mathematics Network

International Mathematical Talent Search

Try your hand at these Mathematical Talent Search problems and mail in your answers! You don't have to have solved all the problems.

Choose one or more of the rounds below, solve as many of the problems as you can in that round (you need not solve them all), and mail your solutions to:

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Make sure that the front page of your solutions contains your **full name and mailing address**.

These problems are made available through the quarterly journal *Mathematics and Informatics*. Student subscriptions at US$12 (student rate) or US$18 (teacher rate) may be ordered from Prof. George Berzsenyi, Department of Mathematics, Box 121, Rose-Hulman Institute of Technology, Terre Haute, IN 47803-3999, U.S.A. Remittances should be payable to *Mathematics and Informatics*.

Round 16

Problem 1/16. Prove that if $a+b+c=0$, then $a^3+b^3+c^3=3abc$.

Problem 2/16. For a positive integer n, let $P(n)$ be the product of the nonzero base 10 digits of n. Call n "prodigitious" if $P(n)$ divides n. Show that one cannot have a sequence of fourteen consecutive positive integers that are all prodigitious.

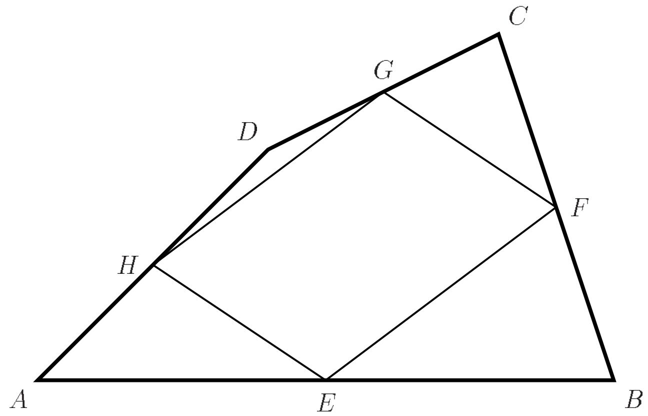
Problem 3/16. Disks numbered 1 through n are placed in a row of squares, with one square left empty. A move consists of picking up one of the disks and moving it into the empty square, with the aim to rearrange the disks in the smallest number of moves so that disk 1 is in square 1, disk 2 is in square 2, and so on until disk n is in square n and the last square is left empty. For example, if the original arrangement is

3	2	1	6	5	4		9	8	7	12	11	10

then it takes at least 14 moves; e.g., we could move the disks into the empty square in the following order: 7, 10, 3, 1, 3, 6, 4, 6, 9, 8, 9, 12, 11, 12.

What initial arrangement requires the largest number of moves if $n=1995$? Specify the number of moves required.

Problem 4/16. Let $ABCD$ be an arbitrary convex quadrilateral, with E, F, G, H the midpoints of its sides, as shown in the figure below. Prove that one can piece together triangles AEH, BEF, CFG, DGH to form a parallelogram congruent to parallelogram $EFGH$.



Problem 5/16. An equiangular polygon $ABCDEFGH$ has sides of length 2, $2\sqrt{2}$, 4, $4\sqrt{2}$, 6, 7, 7, 8. Given that $AB = 8$, find the length of EF.

Round 17

Problem 1/17. The 154-digit number, 19202122... 939495, was obtained by listing the integers from 19 to 95 in succession. We are to remove 95 of its digits, so that the resulting number is as large as possible. What are the first 19 digits of this 59-digit number?

Problem 2/17. Find all pairs of positive integers (m, n) for which $m^2 - n^2 = 1995$.

Problem 3/17. Show that it is possible to arrange in the plane 8 points so that no 5 of them will be the vertices of a convex pentagon. (A polygon is *convex* if all of its interior angles are less than or equal to 180 degrees).

Problem 4/17. A man is 6 years older than his wife. He noticed 4 years ago that he has been married to her exactly half of his life. How old will he be on their 50th anniversary if in 10 years she will have spent two-thirds of her life married to him?

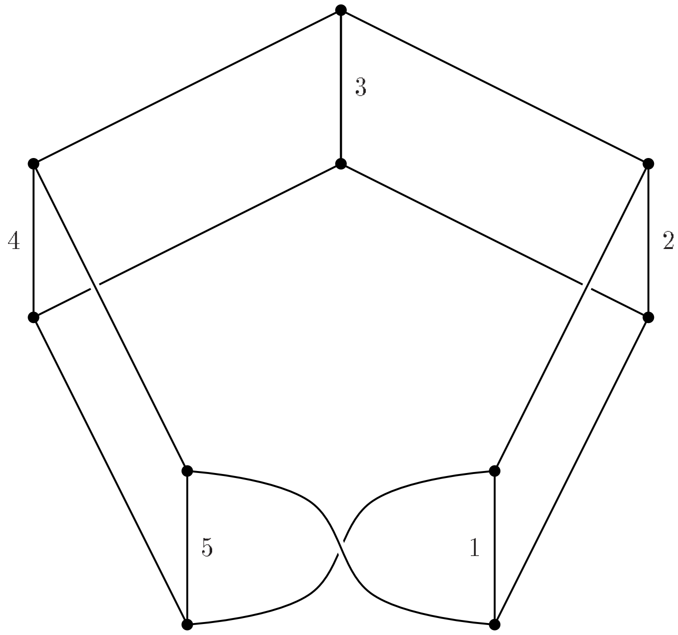
Problem 5/17. What is the minimum number of 3 by 5 rectangles that will cover a 26 by 26 square? The rectangles may overlap each other and/or the edges of the square. You should demonstrate your conclusion with a sketch of the covering.

Round 18

Problem 1/18. Determine the minimum length of the interval $[a, b]$ such that $a \leq x + y \leq b$ for all real numbers $x \geq y \geq 0$ for which $19x + 95y = 1995$.

Problem 2/18. For a positive integer $n \geq 2$, let $P(n)$ denote the product of the positive integer divisors (including 1 and n) of n. Find the smallest n for which $P(n) = n^{10}$.

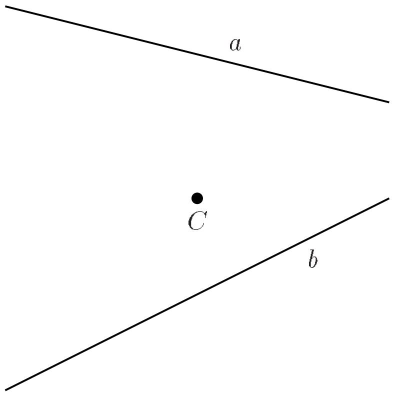
Problem 3/18. The graph shown below has 10 vertices, 15 edges, and each vertex is of order 3 (i.e., at each vertex 3 edges meet). Some of the edges are labelled 1, 2, 3, 4, 5 as shown. Prove that it is possible to label the remaining edges 6, 7, 8, ..., 15 so that at each vertex the sum of the labels on the edges meeting at that vertex is the same.



Problem 4/18. Let a, b, c, d be distinct real numbers such that $a+b+c+d=3$ and $a^2+b^2+c^2+d^2=45$. Find the value of the expression

$$\frac{a^5}{(a-b)(a-c)(a-d)}+\frac{b^5}{(b-a)(b-c)(b-d)}+$$
$$\frac{c^5}{(c-a)(c-b)(c-d)}+\frac{d^5}{(d-a)(d-b)(d-c)}.$$

Problem 5/18. Let a and b be two lines in the plane, and let C be a point, as shown below. Using only a compass and an unmarked straightedge, construct an isosceles right triangle ABC, so that A is on line a, B is on line b, and AB is the hypotenuse of triangle ABC.



Round 19

Problem 1/19. It is possible to replace each of the $\pm$ signs below by either $-$ or $+$ so that

$$\pm 1 \pm 2 \pm 3 \pm 4 \pm \cdots \pm 96 = 1996.$$

At most how many of the $\pm$ signs can be replaced by a $+$ sign?

Problem 2/19. We say that (a, b, c) is a *primitive Heronian triple* if a, b, and c are positive integers with no common factors (other than 1), and if the area of the triangle whose sides measure a, b, and c is also an integer. Prove that if $a = 96$, then b and c must both be odd.

Problem 3/19. The numbers in the 7 by 8 rectangle shown below were obtained by putting together the 28 distinct dominoes of a standard set, recording the number of dots (ranging from 0 to 6) on each side of the dominoes, and then erasing the boundaries among them. Determine the original boundaries among the dominoes. (Note: each domino consists of two adjoint squares, referred to as its sides).

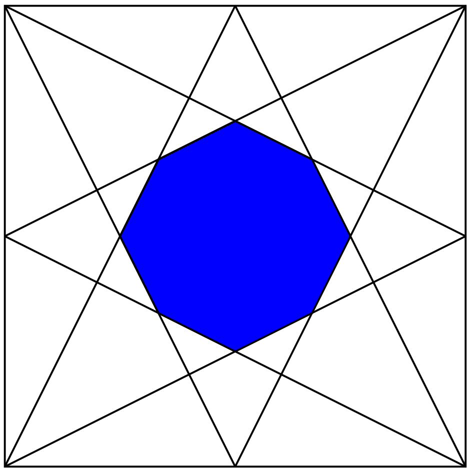
5	5	5	2	1	3	3	4
6	4	4	2	1	1	5	2
6	3	3	2	1	6	0	3
3	0	5	5	0	0	0	6
3	2	1	6	0	0	4	2
0	3	6	4	6	2	6	5
2	1	1	4	4	4	1	5

Problem 4/19. Suppose that f satisfies the functional equation

$$2f(x) + 3f(\frac{2x + 29}{x - 2}) = 100x + 80.$$

Find $f(3)$.

Problem 5/19. In the figure below, determine the area of the shaded octagon as a fraction of the area of the square, where the boundaries of the octagon are lines drawn from the vertices of the square to the midpoints of the opposite sides.



Round 20

Problem 1/20. Determine the number of points (x, y) on the hyperbola

$$2xy - 5x + y = 55$$

for which both x and y are integers.

Problem 2/20. Find the smallest value of n for which the following statement is true: Out of every set of n positive integers one can always choose seven numbers whose sum is divisible by 7.

Problem 3/20. The husbands of 11 mathematicians accompany their wives to a meeting. Sometimes the husbands pass one another in the halls, but once any particular pair have passed each other once, they never pass each other again. When they pass one another, either only one of them recognizes the other, or they mutually recognize each other, or neither recognizes the other. We will refer to the event of one husband recognizing another one as a "sighting", and to the event of them mutually recognizing each other as a "chat", since in that case they stop for a chat. Note that each chat accounts for two sightings.

If 61 sightings take place, prove that one of the husbands must have had at least two chats.

Problem 4/20. Suppose that a and b are positive integers such that the fractions $a/(b-1)$ and a/b, when rounded (by the usual rule; i.e., digits 5 and larger are rounded up, while digits 4 and smaller are rounded down) to three decimal places, both have the decimal value .333.

Find, with proof, the smallest possible value of b.

Problem 5/20. In the figure shown below, the centres of the circles C_0, C_1, and C_2 are collinear, A and B are the points of intersection of C_1 and C_2, and C is a point of intersection of C_0 and the extension of AB. Prove that the two small circles shown, tangent to C_0, C_1 and BC, and to C_0, C_2 and BC, respectively, are congruent to one another.