MAT 1060H1F
Assignment 1

Prof. McCann — Due on Crowdmark noon Thursday Sept. 16

Familiarize yourself with the contents and exercises from Evans Chapter 1 and Appendices A, B and C. (If you like to read ahead, I plan to cover topics from Chapter 2.1-2.2 in the next week or so.)

A continuous function u taking real values on an open set $U \subset \mathbf{R}^n$ is said to have the *mean-value* property if its average over any ball $B_r(x) := \{y \in \mathbf{R}^n \mid |y - x| < r\}$ in U agrees with its value at the center, i.e. for all $r > 0$ and $x \in \mathbf{R}^n$,

$$u(x) = \fint_{B_r(x)} u(y)dy := \frac{1}{\alpha(n)r^n} \int_{B_r(x)} u(y)dy \quad \text{where} \quad \alpha(n) := |B_1(0)| := \int_{B_1(0)} dy.$$

To be handed in:

1. **Linearity** (a) Show the set of functions having the mean value property on U form a vector space.

(b) Find the dimension of this vector space when $U = \mathbf{R}$.

(c) For $n \geq 2$, show the vector space of functions with the mean value property on $U = \mathbf{R}^n$ has dimension strictly greater than $n + 1$. (Can you guess its dimension?)

2. **Harnack inequality**. Show if $u : B_r(0) \longrightarrow [0, \infty)$ has the mean-value property, then there exists a constant $C \leq 2^n$ such that $u(x) \leq Cu(0)$ for all $x \in B_{r/2}(0)$.

3. **Decay of oscillations**. Define $osc_U u := \sup_{x,y \in U}[u(x) - u(y)]$. Use the Harnack inequality to show that any u satisfying the mean value property on $U = B_1(0)$ satisfies

$$osc_{B_{1/3}(0)} u \leq (1 - \frac{1}{2^n}) osc_{B_1(0)} u$$

HINT: If stuck, consult Gilbarg & Trudinger Chapter 8.

4. **Hölder continuity**. Fix a continuous function u on $B_1(0)$, and suppose there exists $\eta < 1$ such that $osc_{B_{r/2}(x)} u \leq \eta \, osc_{B_r(x)} u$ for all balls $B_r(x) \subset B_1(0)$. Use an iterative argument to show there exists $M < \infty$ depending only on η and/or n, such that for any distinct $x, y \in B_{1/2}(0)$,

$$\frac{u(x) - u(y)}{|x - y|^\beta} \leq M \sup_{z \in B_1(0)} |u(z)|$$

where $(\frac{1}{2})^\beta = \eta$. This inequality is often abbreviated as $\|u\|_{C^\beta(B_{1/2}(0))} \leq M\|u\|_{L^\infty(B_1(0))}$.

5. **Liouville theorem**. (a) Use the Harnack inequality to show that any function $u \geq -1$ satisfying the mean-value property on $\mathbf{R}^n$ is bounded above.

(b) Use oscillation decay to show any bounded function satisfying the mean-value property on $\mathbf{R}^n$ is constant.