MAT 1060H1F
Assignment 9

Prof. McCann Due: noon Tuesday Nov. 30

We will proceed through Evans Chapter 5 in class.

To be handed in: Evans (Second edition) # 5.8, 5.11, 5.14, 5.17, 5.18 plus

1. Suppose $\rho(\tau, y)$ and $u(t, x)$ are non-negative functions on $\mathbf{R}^n \times (1, \infty)$ related by

$$\rho(\tau, y) = \frac{1}{\tau^{n/2}} u(\log \tau, \frac{y}{\tau^{1/2}}).$$

(a) Show ρ satisfies the heat equation if and only if u satisfies

$$\frac{\partial u}{\partial t} = \Delta u + \frac{1}{2} \nabla \cdot (xu) =: -Lu.$$

(b) Show the fixed Gaussian $u(t, x) = e^{-x^2/4} =: u_\infty(x)$ satisfies the equation above.

(c) Find the linear operator $L_\theta := u_\infty^{-\theta} L u_\infty^\theta$ governing the evolution $\frac{\partial v_\theta}{\partial t} = -L_\theta v_\theta$ of $v_\theta(t, x) = u_\infty^{-\theta}(x) u(t, x)$.

(d) For $\theta = 1/2$, show the operator L_θ acts symmetrically on smooth compactly supported functions $u, v \in C_c^\infty(\mathbf{R}^n) \subset L^2(\mathbf{R}^n)$, i.e. show

$$\langle u, L_{1/2} v \rangle_{L^2(\mathbf{R}^n)} = \langle L_{1/2} u, v \rangle_{L^2(\mathbf{R}^n)}.$$

(e) When $n = 1$, $L^2(\mathbf{R})$ admits a basis $\{\phi_k\}_{k=0}^\infty$ of eigenfunctions for $L_{1/2}$ with eigenvalues $L_{1/2}\phi_k = k\phi_k$. Use this fact to relate the solution $v_{1/2}(t)$ at time t to its initial data $0 \le v_{1/2}(1) \in C_c^\infty(\mathbf{R})$.

(f) What can you conclude about the rate at which $v_{1/2}$ and the corresponding u approach a multiple of the fixed solutions $u_\infty^{1/2}$ and u_∞, respectively?