Econometrica, Vol. 53, No. 4 (July, 1985)

THE STRATEGY STRUCTURE OF TWO-SIDED MATCHING MARKETS

GABRIELLE DEMANGE AND DAVID GALE[1]

We study two-sided markets in which agents are buyers and sellers or firms and workers or men and women. The agents are to form partnerships (which provide them with satisfaction) and at the same time make monetary transfers (e.g. salaries or dowries). The core of this market game is shown to have a particularly nice structure so that precise answers can be given to questions concerning comparative statics and manipulability.

1. INTRODUCTION

THE IDEA OF USING Walrasian equilibrium as a mechanism for making allocations with desirable properties of fairness and efficiency has been studied extensively. The scheme involves having agents specify their supply and demand functions. The competitive equilibria are then calculated and allocations are made accordingly. In general this procedure may run into two difficulties: first, nonuniqueness—if there are several equilibria there may be no fair way to decide which one should be implemented; and second, manipulability—if there is only one equilibrium an informed agent may be able to influence it by suitably falsifying his demand data. However, as we will show here, using the Walrasian mechanism does work remarkably well for a certain class of important markets. These are the "matching markets" of our title and they include markets for single items, like houses, where it is assumed that traders do not wish to acquire more than one item. They also include labor markets in which it is desired to match workers with jobs at suitable salaries, "academic markets" in which students are to be assigned to educational institutions, "marriage markets" where men and women are matched through negotiating of dowries. It turns out that in these markets both the problems of nonuniqueness and manipulability can be resolved in a very precise manner which we will now describe.

We consider first the following situation. A set of objects is to be distributed among a set of buyers under the condition that no buyer is interested in acquiring more than one object. It is assumed that given a vector of prices for the objects each buyer is able to specify which object or set of objects (possibly empty) he wishes to buy. This is the usual buyer's *demand function*. Further, at the given prices the seller or sellers specify which objects they are willing to sell (their *supply function*). One now asks the usual economic question, do there always exist equilibrium prices, i.e., prices at which every buyer gets an object in his demand set, every seller who wishes to sell is able to do so, and no two buyers get the same object.

A special case of this market structure has been investigated in previous work (Gale [4], Shapley-Shubik [11], and Demange [2]). This is the "linear case" in which each buyer and seller places a monetary value on each of the objects.

[1] The work of the first author was supported in part by Le Commissariat au Plan, France.

Buyers then demand the object or objects which maximize the excess of their valuations over the announced prices, assuming this excess to be nonnegative, and sellers will want to sell if the announced price exceeds their valuation. For this case the existence of equilibrium prices has been known for some time (see, e.g., [**4**]).

For general demand structures a proof was sketched by Crawford and Knoer in [**1**]. A complete proof using different techniques has been given by Quinzii [**9**] and a proof using combinatorial topology has been given by Gale [**5**] under even more general conditions on demand (e.g., whether a buyer prefers object A to object B may depend on the price of a third object, C).

In this paper we investigate further the set of all equilibrium prices which turns out to have some rather striking properties which are not present in the usual exchange models. Our first results show that the set of equilibrium price vectors form a lattice, so, in particular, there exists a smallest and largest equilibrium price vector. This was shown in [**10**] for the linear case and the existence of smallest and largest price equilibria, but not the lattice property, also follows from the arguments sketched in [**1**].

The main portion of the paper is devoted to analyzing the properties of the minimum price equilibrium looked upon as a mechanism for allocating the objects. We may imagine, for example, that all traders in the market present their supply-demand functions to a "referee" who then calculates the minimum equilibrium and allocates the objects accordingly. This mechanism can then be thought of as a game in which an agent's strategy consists of announcing a demand function, the payoff being then the corresponding allocation. Our first main result states that for the buyers this mechanism is *nonmanipulable* in the strong sense that no coalition of buyers by falsifying demands can achieve higher payoffs to all of its members.[2] In particular, there is nothing to be gained for a buyer by trying to acquire information on the preferences of other traders since there is no way he can make effective use of such information. This is in sharp contrast to the situation in usual exchange models where, as is well known, an informed trader by suitably falsifying his preferences may be able to obtain a better payoff for himself at the expense of the other traders. Our result is also rather surprising if one thinks about the way discrete markets usually operate. In a housing market for example, the usual allocation mechanisms involve pairwise negotiations between buyers and sellers with offers and counter-offers, and it is usually considered important for the buyers to conceal rather than reveal their true valuations.

Our second result concerns the strategic situation for the sellers which turns out to be quite different from that of the buyers. Indeed, except for very special cases, it will always be possible for sellers to increase their payoff by suitable falsification. Again we are able to give a precise description of sellers' behavior. We show that by specifying their supply functions appropriately the sellers can force the payoff to be given by the maximum rather than the minimum equilibrium

[2] Demange [**2**] and Leonard [**8**] have shown that for the linear case the mechanism is "individually" nonmanipulable.

price. These strategies yield a Nash equilibrium.[3] Further, if any set of strategies does not give the maximum equilibrium price allocation then either some seller is using a dominated strategy or there is at least one seller who can increase his payoff by changing his strategy.

Actually, our results apply to a more general situation than that described above. It might be, for example, that sellers discriminate, being willing to offer a better price to some buyers than to others. A more natural context for such a model is that of workers and firms. Given any two jobs j and j', a worker knows whether he prefers to work at j for salary s or j' for salary s'. Similarly, firms are able to determine which workers they prefer to hire at which salaries. The problem now is no longer one of equilibrium but rather a question about the core of a certain game. One seeks an assignment of workers to jobs at specified salaries which is *stable*, meaning that under the given assignment no pair consisting of a worker and firm can make an employment-salary arrangement which is more satisfactory to both than the given one.[4] All the results previously described extend to this more general model. The stable assignments form a lattice and the worker-optimal stable assignment is nonmanipulable by the workers. Thus the workers should reveal their true demands to the referee (who should perhaps be called.the arbitrator in this context). On the other hand they must avoid revealing their true demands to the firms, since according to our second result the firms could make use of this information to force the payoff to be firm rather than worker-optimal.

We remark that results precisely analogous to those presented here have been proved in [**7**, **5**] (the lattice property), and [**3**, **9**] (nonmanipulability) for a model without money. A typical application for such models is, for example, the problem of assigning students to universities. The present model also would apply in that context if one introduced the possibility of financial aid of different amounts. However, the model of [**6**] is not a special case of our model because in [**6**] it has to be assumed that all preferences of agents are strict, e.g., a student cannot be indifferent between universities, and similarly for the universities. In our model, on the other hand, it is essential that indifference can always be achieved by means of suitable transfers. This is, of course, a rather strong assumption. In the worker-employer application we must assume that a worker will accept any job provided the salary is sufficiently high, and an employer will hire any worker provided he is willing to accept a sufficiently low, possibly negative, salary. For such a model to be reasonable we must suppose that all of the workers are at least minimally qualified for all of the jobs under consideration. This is the usual assumption in this sort of assignment problem.

There is another somewhat fanciful case of our model in which the participants are men and women and the assignments consist of (monogamous) marriage contracts with dowries (positive or negative). This model is completely symmetric since both directions of transfer payments are possible.

Finally, we should point out a limitation on the generality of our results on nonmanipulability. We must suppose that there can be no monetary transfers

[3] This statement must be modified slightly; see Sections 2 and 6.

[4] We wish to thank W. Thomson for suggesting to us the study of this problem.

between traders on the same side of the market. This as well as our other results will be illustrated by the examples of the next section. The formal analysis is presented in the final three sections.

2. EXAMPLES

In its simplest form our results are familiar. This is the case where there is only one object. The allocation mechanism then corresponds to well known auction models. Each buyer (who should now be called a bidder) and the seller submit their valuations of the object to the referee (who should now be called the auctioneer). Suppose the valuations of the buyers are $u_1 \geqslant u_2 \cdots \geqslant u_n$ and that of the seller is v. One easily sees that p is an equilibrium price if and only if $u_1 \geqslant p \geqslant \max[u_2, v]$ (by convention if $v > u_1$ we set $p = v$ and there is no transaction). The minimum equilibrium price corresponds exactly to the usual direct or English auction. It is optimal and nonmanipulable for the bidders, for it is easy to see that a bidder cannot help and may hurt himself by falsifying his valuation either up or down. On the other hand, if the seller knows the value u_1 he can pretend his valuation is $u_1 - \varepsilon$, thus forcing the price p arbitrarily close to the upper limit u_1. The ε is necessary here since if the seller sets v equal to u_1 the buyer would be indifferent between buying or not and might choose not to buy. This is the reason for the footnote of the previous section.

As mentioned in the introduction, our nonmanipulability result no longer holds if we allow monetary transfers between bidders. Suppose $u_1 \geqslant u_2 > v$. Then the highest bidder B_1 could bribe the low bidder B_2 to change his bid to some $u_2' \leqslant v$. The price p then becomes v rather than u_2 and B_1 would gain the amount $u_2 - v$, a part of which he could pay to B_2.

In the same model if the auctioneer uses the mechanism of maximum equilibrium price, we have $p = u_1$. This is the so-called Dutch auction which is optimal and nonmanipulable for the sellers. On the other hand if the highest bidder knows the value of $\tilde{v} = \max[u_2, v]$ he should bid $\tilde{v} + \varepsilon$. Again the ε is necessary, for if two bidders both selected the same u_1 the object might go to the honest bidder B_2 rather than the liar B_1.

As a next example consider the case of a single buyer but many objects. Let u_j and v_j be the values placed on the objects by the buyer and sellers respectively. We make the assumption of linear preferences so at prices p the buyer will choose the object for which $u_j - p_j$ is a maximum (we suppose $u_j > v_j$ for at least one object). The minimum equilibrium price vector $\underline{p}$ is then $\underline{p}_j = v_j$ for all j, and the buyer gets the object for which $u_j - v_j$ is a maximum. Assuming $u_1 - v_1 \geqslant u_2 - v_2 \geqslant \cdots$, the maximum equilibrium price is given by $\bar{p}_j = v_j$ for $j > 1$ and $\bar{p}_1 = u_1 - u_2 + v_2$, which make the buyer indifferent between buying the first or second object. Further, the seller of the first object can sell the object for $v_1' = u_1 - u_2 + v_2 - \varepsilon$ by specifying v_1' as his valuation. If the seller-optimal allocation is to be implemented, then the buyer should pretend his value for the first object is $u_1' = u_2 + v_1 - v_2$.

A somewhat more complicated case (see Vickrey [**12**]) is that in which there are several buyers and objects but all the objects are identical. In this case the prices of all objects which are sold must be equal (otherwise the buyer of the more expensive object could pair with the seller of the cheaper one to the advantage of both). Let u_j and v_j be the valuation of objects by buyers and sellers respectively. One sees that the equilibrium prices are precisely the closure of the set of p such that the number of u_i greater than p is the same as the number of v_j less than p. It is an instructive exercise to convince oneself that the upper and lower end points of this interval are nonmanipulable by sellers and buyers respectively.

Finally, we consider some linear 2×2 examples. The entry a_{ij} below gives the value of object i to buyer j. We assume the minimum acceptable selling prices are zero.

	B_1	B_2
0_1	4	2
0_2	1	3

In this example, the two buyers prefer different objects, so the minimum equilibrium prices is $\underline{p} = (0, 0)$. The maximum equilibrium price is $\bar{p} = (4, 3)$ and the sellers can force this by pretending that $\bar{p}$ is their reservation price. The lattice of all equilibrium prices is plotted in Figure 1.

In the final example, both buyers prefer the same object.

	B_1	B_2	
0_1	3	4	3
0_2	1	3	2

In every equilibrium, B_1 gets 0_1, B_2 gets 0_2. The minimum price is $\underline{p} = (1, 0)$, the maximum is $\bar{p} = (3, 2)$ and the price lattice is plotted above. Note that in order to force the price to $\bar{p}$, it is only necessary that the reservation price of 0_1 be set at 3 and that of 0_2 any number less than 2.

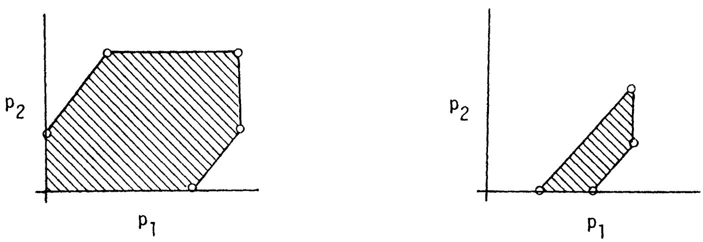

FIGURE 1

3. THE MARKET MODEL

A two-sided market M involves two finite sets P and Q of *agents*. A *matching* μ is a disjoint set of pairs (p, q) in $P \times Q$. We shall write $(p, q) \in \mu$ or $p = \mu(q)$ or $q = \mu(p)$ interchangeably. The structure of agents' preferences is given by *utility functions* in a standard manner. Let $u_{pq}(x)$ be the utility to p of being matched with q and receiving a payment of x. We define v_{pq} analogously for q in Q. The functions u_{pq} and v_{pq} are assumed to be continuous and increasing on R. We further suppose that for each p and q the utility of being unmatched is given by some numbers r_p, s_q. An important further assumption is:

(3.1) The range of u_{pq} and v_{pq} is all of R.

In words (3.1) asserts that no matching is infinitely good or infinitely bad. That is, any agent would be willing to be matched with any other if the compensation was sufficiently high and would refuse such a pairing if it was sufficiently low. For our purposes, it turns out to be more convenient to work with the inverses of utility functions, the *compensation function*, defined by $f_{pq} = u_{pq}^{-1}$, $g_{pq} = v_{pq}^{-1}$. Thus, $f_{pq}(u)$ is the amount of money p must receive in order to achieve utility u if he is matched with q. The f's and g's are also continuous and increasing from R to R. We will denote the array of functions f_{pq} and g_{pq} by f and g. The market is then denoted by $M = (P, Q; f, g; r, s)$.

DEFINITION: A *feasible payoff* (u, v) of M consists of a p-vector u and q-vector v such that there exists a matching μ with the properties:

(3.2) $u_p = r_p (v_q = s_q)$ if $p(q)$ is unmatched;

(3.3) if $(p, q) \in \mu$, then $f_{pq}(u_p) + g_{pq}(v_q) \leq 0$.

The pair (u, v) represents achievable utility levels for P and Q. Condition (3.2) is essentially the definition of r_p and s_q. Condition (3.3) is the requirement that the compensation to the two members of a matched pair is in fact a transfer; what one partner gains, the other loses. More precisely, a pair cannot gain money by being matched (formally we are allowing money to "evaporate" but this is just a mathematical convenience and plays no role in the model). We say the matching μ is *compatible* with the payoff (u, v). Our key notion is given by the following definition.

DEFINITION: The feasible payoff (u, v) is *stable* if

(3.4) $u_p \geq r_p, \qquad v_q \geq s_q,$

(3.5) $f_{pq}(u_p) + g_{pq}(v_q) \geq 0 \qquad$ for all (p, q).

This is the usual stability condition. Condition (3.4) is "individual rationality". If condition (3.5) did not hold then from the fact the f_{pq} and g_{pq} are continuous and increasing one could choose $u'_p > u_p$ and $v'_q > v_q$ such that $f_{pq}(u'_p) + g_{pq}(v'_q) \leq$

0, producing a feasible arrangement giving higher utility to both p and q, so that the pair (p, q) would block the payoff (u, v).

In [**1**, **5**, and **9**] it is proved that for the market models described here there always exists stable payoffs. We will make frequent use of this existence theorem in the following sections. The set of all stable payoffs is called the *core* of M.

4. STRUCTURE OF THE CORE

In this section we derive the essential properties of the core which are of interest in themselves and are needed for the later analysis.

If μ is a matching in $P\times Q$ and $P'\subset P$, $Q'\subset Q$, we will say *P' and Q' are matched by μ* if: (i) $p\in P'$ and p is matched implies $\mu(p)\in Q'$ and (ii) $q\in Q'$ and q is matched implies $\mu(q)\in P'$.

LEMMA 1 (Decomposition): *Let (u^1, v^1) and (u^2, v^2) be payoffs satisfying* (3.3) *and* (3.5) *and let μ^1 and μ^2 be compatible matchings.*

Let $P^1=\{p|u_p^1>u_p^2\}$, $P^2=\{p|u_p^2>u_p^1\}$, $P^0=\{p|u_p^1=u_p^2\}$. Define Q^1, Q^2, Q^0 analogously. Then both μ_1 and μ_2 match P^1 with Q^2 and P^2 with Q^1.

PROOF: *If $p\in P_1$ and $q=\mu^1(p)$*: Then since (u^1, v^1) satisfies (3.3) we have:

$$f_{pq}(u_p^1)+g_{pq}(v_q^1)\leqslant 0.$$

On the other hand since (u^2, v^2) satisfies (3.5) we have

$$f_{pq}(u_p^2)+g_{pq}(v_q^2)\leqslant 0,$$

but $u_p^1>u_p^2$ since $p\in P^1$ so $v_q^1<v_q^2$ since f_{pg} and g_{pq} are increasing; hence $q\in Q^2$, so μ^1 matches P^1 with Q^2.

A symmetrical argument choosing $q\in Q^2$ shows that μ^2 matches P^1 and Q^2. The other assertion is proved by symmetry between (u^1, v^1) and (u^2, v^2).

Q.E.D.

COROLLARY 1: *If (u^1, v^1) and (u^2, v^2) are stable payoffs, then μ^1 and μ^2 are both bijections between P^1 and Q^2, between P^2 and Q^1, and they match P^0 with Q^0.*

PROOF: For p in P^1, $u_p^1>u_p^2\geqslant r_p$ so p is matched (from (3.2)) by μ_1; thus $\mu^1(P^1)\subset Q^2$ and symmetrically $\mu^2(Q^2)\subset P^1$. Since μ^1 and μ^2 are injective (and P^1 and Q^2 are finite) μ^1 and μ^2 are bijections between P^1 and Q^2. The last assertion follows from the injectivity of μ^1 and μ^2. *Q.E.D.*

We derive now a first property of the core:

PROPERTY 1: (A) *If $u_p^1>r_p$ for some stable (u^1, v^1), then p must be matched for every stable payoff.* (B) *If p is matched for some stable payoff, p may be matched for every stable payoff.* (C) *If p is not matched for some stable payoff, p may be not matched for every stable payoff.*

PROOF: For (A) let μ^1 be a matching compatible with (u^1, v^1). Suppose for some stable (u^2, v^2) and μ^2 compatible with (u^2, v^2) p is not matched so that $u_p^2 = r_p$. Now $p \in P^1$ since $u_p^1 > u_p^2$ but by Corollary 1, μ^2 is a bijection between P^1 and Q^2, contradicting the assumption that p is unmatched by μ^2.

For (B) and (C) let μ^1 be compatible with a stable payoff (u^1, v^1); let (u^2, v^2) be another stable payoff and μ^2 compatible with (u^2, v^2). By Corollary 1, we may define the matching $\tilde{\mu}^2$ by: P^1 and Q^2, P^2 and Q^1 are matched by μ^2, P^0 and Q^0 are matched by μ^1. $\tilde{\mu}^2$ is obviously compatible with (u^2, v^2) and matches the same agents as μ^1. So for (B) take μ^1 such that p is matched and for (C) μ^1 such that p is not matched. *Q.E.D.*

We wish to study the core of M as a function of r and s. We will use the notation $M(r, s)$ for the market where r and s may vary but P, Q, f, g are fixed.

LEMMA 2: *Let (u^1, v^1) be stable for $M(r^1, s^1)$ and let (u^2, v^2) be stable for $M(r^2, s^2)$ where $r^1 \leqslant r^2 \leqslant u^1$ and $s^1 \leqslant s^2$. Then*:

(4.1) $(\bar{u}, \underline{v}) = (u^1 \vee u^2, v^1 \wedge v^2)$ *is stable for* $M(r^1, s^1)$;

(4.2) $(\underline{u}, \bar{v}) = (u^1 \wedge u^2, v^1 \vee v^2)$ *is stable for* $M(r^2, s^2)$.

PROOF: Define P^2 and Q^1 as in Lemma 1 and let μ^1 and μ^2 be associated matchings of (u^1, v^1) and (u^2, v^2). Since for p in P^2 $u_p^2 > u_p^1 \geqslant r_p^2$, all of P^2 is matched by μ^2 and all of Q^1 is matched by μ^1 so by Corollary 1 μ^1 and μ^2 are bijections between P^2 and Q^1. To prove (4.1) define the matcing $\bar{\mu}$ by:

P^2 and q^1 are matched by μ^2;

$P - P^2$ and $Q - Q^1$ are matched by μ^1.

We must show that this is compatible with $(\bar{u}, \underline{v})$. Condition (3.3) follows since it holds for ^{1} and ^{2}. Condition (3.4) follows since $r^1 \leqslant r^2$ and $s^1 \leqslant s^2$. Condition (3.5) is also immediate if $\bar{u}_p = u_p^i$ and $v_q = v_q^i$, $i = 1, 2$. If, say, $\bar{u}_p = u_p^1$ and $\underline{v}_q = v_q^2$, then

$$f_{pq}(\bar{u}_p) + g_{pq}(\underline{v}_q) \geqslant f_{pq}(u_p^2) + g_{pq}(v_q^2) \geqslant 0$$

by (3.5) applied to (u^2, v^2), etc. . . .

To prove (4.2) if p is unmatched by $\bar{\mu}$, then $p \in P - P^2$ and is unmatched by μ^1 so $\bar{u}_p = u_p^1 = v_p^1$. If q is unmatched by $\bar{\mu}$, then $q \in Q - Q^1$ and $\underline{v}_q = v_q^1$, so q is unmatched by μ^1, so $\underline{v}_q = s_q^1$.

To prove (4.2) define $\underline{\mu}$ by

P^2 and Q^1 are matched by μ^1;

$P - P^2$ and $Q - Q^1$ are matched by μ^2.

The arguments for (3.3) and (3.5) are as before. For (3.4) $u_p^1 \geqslant r_p^2$ by hypothesis and $u_p^2 \geqslant r_p^2$ by (3.4), and $\bar{v}_q \geqslant v_q^2 \geqslant s_q^2$. To prove (3.2), if p is unmatched by $\underline{\mu}$, then $p \in P - P^2$, so $\underline{u}_p = r_p^2$ and p is unmatched by μ^2, so $\underline{u}_p = v_p^2$. If q is unmatched by $\underline{\mu}$, then $q \in Q - Q^1$, so $\bar{v}_q = v_q^2$ and q is unmatched by μ^2, so $\bar{v}_q = s_q^2$. *Q.E.D.*

Define $U_M = \{u \in R^P | (u, v)$ in M-stable for some $v\}$.
Define $V_M = \{v \in R^Q | (u, v)$ in M-stable for some $u\}$.

PROPERTY 2: *The sets U_M and V_M are lattices with smallest and largest elements.*

PROOF: The lattice property follows from Lemma 2 in the special case where $r = r^1$, $s = s^1$. It is easily seen from the continuity of f and g that U_M and V_M are closed sets and hence have smallest and largest elements. *Q.E.D.*

If $\bar{u}$ in the largest element of U_M and $\underline{v}$ in the smallest element of V_M, we call $(\bar{u}, \underline{v})$ the P-optimal payoff. Analogously $(\underline{u}, \bar{v})$ is the Q-optimal payoff. We now derive a result of comparative statics.

PROPERTY 3: *If $(\bar{u}(r, s), \underline{v}(r, s))$ is the P-optimal payoff in $M(r, s)$ then $\bar{u}(r, s)$, $(\underline{v}(r, s))$ is an increasing (decreasing) function of $r(s)$ and $\underline{v}(r, s)$ is a decreasing (increasing) function of $r(s)$.*

PROOF: Suppose $r' \geq r$ and (u', v') is stable in $M(r', s)$. Then from (4.2) (interchanging P and Q and taking $s' = s$), we have $(u' \vee \bar{u}(r, s)),\ v' \wedge \underline{v}(r, s))$ is stable for $M(r', s)$ so $\bar{u}(r', s) \geq \bar{u}(r, s)$ and $\underline{v}(r', s) \leq \underline{v}(r, s)$. Suppose $s' \geq s$ and (u, v) is stable in $M(r, s')$. Then from (4.1) $(u \vee \bar{u}(r, s'),\ v \wedge \underline{v}(r, s'))$ is stable for $M(r, s)$ so $\bar{u}(r, s) \geq \bar{u}(r, s')$ and $\underline{v}(r, s) \leq \underline{v}(r, s')$. *Q.E.D.*

As a special case of Property 3 we have:

COROLLARY 3: *If additional agents of type Q enter the market, then $\bar{u}$ does not decrease and $\underline{v}$ does not increase. If additional agents of type P enter the market, then $\bar{u}$ does not increase and $\underline{v}$ does not decrease.*

PROOF: This is a special case of Property 3, for if s_q is sufficiently large then from condition (3.1) q will never be matched by any feasible matching. More precisely choose s_q so that $g_{pq}(s_q) > -f_{pq}(r_p)$ for all p in P. Then (3.3) and (3.4) can never be satisfied so any stable payoff will be the same for P and $Q - q$ as it would be if q were not in the market. A symmetrical argument applies for p in P. *Q.E.D.*

PROPERTY 4: *The function $\bar{u}(r, s)$ $(\underline{v}(r, s))$ is upper (lower) semi-continuous.*

PROOF: Let (r'', s'') approach (r, s) and let $(\bar{u}(r^n, s^n)\underline{v}(r^n, s^n))$ approach $(\tilde{u}, \tilde{v})$. If μ'' is compatible with $(\bar{u}(r^n, s^n),\ \underline{v}(r^n, s^n))$, then μ'' must take on some value $\tilde{\mu}$ infinitely often since there are only a finite number of matchings. It now follows from continuity of f and g that $\tilde{\mu}$ is compatible with $(\tilde{u}, \tilde{v})$, so $(\tilde{u}, \tilde{v})$ is stable in $M(r, s)$, so $\tilde{u} \leqslant \bar{u}(r, s)$ and $\tilde{v} \geqslant \underline{v}(r, s)$. *Q.E.D.*

The following property of the P-optimal payoff will be needed in the next section.

LEMMA 3: *If* $|P| \leqslant |Q|$, *then* $\underline{v}_q = s_q$ *for some* q *in* Q.

PROOF: The proof is immediate unless $|P| = |Q|$ and all of P is matched under $(\bar{u}, \underline{v})$. Suppose now $\underline{v} > s$. It follows from Lemma 2 (4.1) that $(\bar{u}, \underline{v})$ remains a P-optimal payoff when r is replaced by $\bar{u}$, i.e., for $M' = (P, Q, f, g, \bar{u}, s)$. We may now replace r by $\tilde{r} = \bar{u} + \varepsilon$, $\varepsilon > 0$, getting a market $M = (P, Q, g; \tilde{r}, s)$ and from Property 4 for ε sufficiently small the P-optimal payoff $(\tilde{u}, \underline{v})$ will have $\underline{v} > s$. Further, $\tilde{u} \geqslant \bar{u} + \varepsilon > \bar{u}$. But since $\underline{v} > s$, all of Q must be matched, hence all of P is matched under $(\tilde{u}, \underline{v})$. Now note that $(\tilde{u}, \underline{v})$ is still stable if $\tilde{r}$ is replaced by any $r' \leqslant \tilde{r}$, in particular if $r' = r$, but since $\tilde{u} > u$ this contradicts P-optimality of $(\bar{u}, \underline{v})$. *Q.E.D.*

We will need one more property of P-optimal payoffs. We say a pair (p, q) is *compatible* for payoff (u, v) if $f_{pq}(u_p) + g_{pq}(v_q) = 0$.

LEMMA 4: *Suppose* $\underline{v}_q > s_q$ *for* $q \in Q' \subset Q$ *and let* $P' = \mu(Q')$. *Then there is a compatible pair* (p, q), $p \in P - P'$, $q \in Q'$.

PROOF: Note from Lemma 3, $P - P' \neq \varnothing$. Arguing by contradiction, suppose $f_{pq}(\bar{u}_p) + g_{pq}(\underline{v}_q) > 0$ for all $p \in P - P'$, $q \in Q'$. Then for some positive ε,

$$f_{pq}(\bar{u}_p) + g_{pq}(\underline{v}_q - \varepsilon) > 0 \quad \text{for} \quad p \in P - P', q \in Q'. \tag{4.3}$$

Let $M' = (P', Q'; f, g; r, s')$ where $s'_q = \underline{v}_q - \varepsilon$, and let $(\bar{u}', \underline{v}')$ be the P-optimal payoff for M'. By Lemma 3, $v'_q = \underline{v}_q - \varepsilon$ for some q in Q'. We claim the payoff $(\tilde{u}, \underline{v})$ is stable where

$$\begin{aligned} \tilde{u}_p &= \bar{u}_p \quad \text{for} \quad p \in P - P', \\ &= \bar{u}'_p \quad \text{for} \quad p \in P'; \\ \underline{v}_q &= \underline{v}'_q \quad \text{for} \quad q \in Q', \\ &= \underline{v}_q \quad \text{for} \quad q \in Q - Q'. \end{aligned}$$

In fact, the only possible unstable pair (p, q) must have $p \in P - P'$, $q \in Q'$ but this is not possible because of (4.3). *Q.E.D.*

5. NONMANIPULABILITY BY THE AGENTS OF TYPE P

The following result shows that the P-optimal payoff is Pareto optimal among all feasible payoffs for the agents of type P.

THEOREM 1 (Pareto Optimality): *Let $(\bar{u}, \underline{v})$ be the P-optimal payoff and let (u, v) be any Q-individually rational feasible payoff. Then it is not the case that* $u > \bar{u}$.

PROOF: Let μ be a matching corresponding to (u, v). If $u > \bar{u}$, then $u > r$, so that all of P is matched by μ, so $v_q < \underline{v}_q$ for q in $\mu(P)$, so $\underline{v}_q > s_q$ for $|P|$ members of Q, so all of P is matched under $(\bar{u}, \underline{v})$ by some $\bar{\mu}$, but by Lemma 3, we have $\underline{v}_q = s_q$ for some q in $\bar{\mu}(P)$, so we cannot have $v_q < \underline{v}_q$ by q rationality of (u, v). *Q.E.D.*

Our key lemma is the following:

LEMMA 5: *Let (u, v) be a feasible payoff with matching μ and let $P^+ = \{p \in P | u_p > \bar{u}_p\}$. Then there is a p in $P - P^+$ and q in $\mu(P^+)$ such that* $f_{pq}(u_p) + g_{pq}(v_q) < 0$.

PROOF: Let $\bar{\mu}$ be a matching compatible with $(\bar{u}, \underline{v})$. There are two cases.

CASE 1: $\mu(P^+) \neq \bar{\mu}(P^+)$. Since $u_p > \bar{u}_p \geqslant r_p$ for p in P^+ it follows that P^+ is matched by μ. Choose $q \in \mu(P^+)$, $q \notin \bar{\mu}(P^+)$ say, $q = \mu(p')$. Since $u_{p'} > \bar{u}_{p'}$ it follows from Lemma 1 that $v_q < \underline{v}_q$, hence q was matched under $\bar{\mu}$, say $q = \bar{\mu}(p)$ where $p \in P - P^+$. By feasibility, $f_{pq}(\bar{u}_p) + g_{pq}(\underline{v}_q) \leqslant 0$ but $u_p \leqslant \bar{u}_p$ since $p \in P - P^+$ and $v_q < \underline{v}_q$, so the assertion is proved since f and g are increasing.

CASE 2: $\mu(P^+) = \bar{\mu}(P^+)$. Since $u_p > \bar{u}_p$ for all p in P^+, it follows that $v_q < \underline{v}_q$ for all q in $\mu(P^+)$. By Lemma 4, there is p in $P - P^+$ and q in $\mu(P^+)$ such that $f_{pq}(\bar{u}_p) + g_{pq}(\underline{v}_q) = 0$ and the result follows since $v_q < \underline{v}_q$ and $u_p \leqslant \bar{u}_p$. *Q.E.D.*

We now consider the possibility of agents falsifying their utilities by manipulating the f_{pq}'s or r_p's.

DEFINITION: Let P' be any subset of P and let $M' = (P, Q; f', g; r', s)$ where $f'_p = f_p$ and $r'_p = r_p$ for $p \notin P'$. Suppose (u', v') is any feasible payoff for M'. The *true utility* $\tilde{u}_p$ for p in P' under (u', v') is defined by $\tilde{u}_p = f_{pq}^{-1} f'_{pq}(u_p)$ if p is matched, $\tilde{u}_p = r_p$ if p is not matched.

THEOREM 2 (Nonmanipulability): *Let (u', v') be any stable payoff for the market M' and let $(\bar{u}, \underline{v})$ be the P-optimal payoff for M. Then $\bar{u}_p \geqslant \tilde{u}_p$ for at least one p in* P'

PROOF: Suppose $\tilde{u}_p > \bar{u}_p$ for all p in P'. We claim $(\tilde{u}, v')$ is feasible for M, for if μ' is a matching for (u', v') and $(p, q) \in \mu'$, then

$$\begin{aligned} f_{pq}(\tilde{u}_p) + g_{pq}(v'_q) &= f_{pq}(f_{pq}^{-1}(u'_p))) + g_{pq}(v'_q) \\ &= f'_{qp}(u'_p) + g_{pq}(v'_q) \leqslant 0. \end{aligned}$$

Now let $P^+=\{p|\tilde{u}_p>\bar{u}_p\}$. From Theorem 1, $P-P^+$ is nonempty and from Lemma 5 there is $p\in P-P^+$ and $q\in\mu(P^+)$ such that:

$$f_{pq}(\tilde{u}_p)+g_{pq}(v'_q)<0.$$

But since $p\notin P'$, we have $\tilde{u}_p=u'_p$ and $f_{pq}=f'_{pq}$, hence

$$f'_{pq}(u'_p)+g_{pq}(v'_q)<0,$$

contradicting stability of (u', v'). *Q.E.D.*

6. THE MANIPULATION BY THE AGENTS OF TYPE Q[3]

As indicated by the examples of Section 2, the P-optimal payoff is almost always manipulable by the agents of type Q. It is therefore natural to try to predict the result of such manipulation assuming the Q agents have unlimited information and possibilities for communication with each other. One seeks some kind of joint strategy which these agents might be expected to follow. A natural requirement to impose on such a joint strategy is that once it is adopted, there should be no incentive for any individual or group of individuals to depart from it. In other words, one is asking for the existence of a strong Nash equilibrium. In this section, we show that such equilibria exist, that they all achieve the same payoff, namely $\bar{v}(M)$ for the Q agents, and finally that these are the only Nash equilibria (aside from rather unnatural cases in which an agent uses a dominated strategy) when the Q agents manipulate only their reservation prices.

We consider the following mechanism: the Q agents may replace their functions g and constants s by any γ and σ. An outcome is then an allocation, i.e., a matching plus transfers between matched agents, leading to the P-optimal payoff of the market $M=(P, Q; f, \gamma; r, \sigma)$. Since P, Q, f, r are fixed we denote such a market by $M(\gamma, \sigma)$ and the P-optimal payoff by $(\bar{u}(\gamma,\sigma), \underline{v}(\gamma,\sigma))$. Thus if μ is the matching compatible with this payoff which is chosen, the transfers to Q agents are: $\gamma_{pq}(\underline{v}_q(\gamma,\sigma))$ if (p, q) is in μ and 0 if q is not matched. And thus the true payoff $\tilde{v}^\mu(\gamma,\sigma)$ is:

$$\begin{aligned}\tilde{v}^\mu_q(\gamma,\sigma)&=g^{-1}_{pq}(\gamma_{pq}(\underline{v}_q(\gamma,\sigma)) \quad \text{if} \quad (p,q) \text{ is in } \mu,\\ &=s_q \quad \text{if } q \text{ is not matched.}\end{aligned}$$

It is noteworthy that when there are several matchings compatible with the P-optimal payoff, the true payoff of an agent q may depend on the one selected but it is not the case when he does not lie, i.e., when $(g_q, s_q)=(\gamma_q,\sigma_q)$. Once a compatible matching is chosen, the transfers and true payoff are completely determined so we say for short that μ is an outcome of (γ,σ) keeping in mind that some transfers are associated with it.

We say that (γ,σ,μ) is a *strong equilibrium* if μ is an outcome of (γ,σ) and if there is no subset $Q'\subset Q$ and pair (γ',σ'), μ', where μ' is an outcome of (γ',σ'), $(\gamma'_q,\sigma'_q)=(\gamma_q,\sigma_q)$ for $q\in Q-Q'$ and $\tilde{v}^{\mu'}_q(\gamma',\sigma')>\tilde{v}^\mu_q(\gamma,\sigma)$ for $q\in Q'$.

THEOREM 3: *The Q-vector* $\bar{v}$ *is a strong equilibrium payoff.* (*Here,* $\bar{v}$ *is the Q-optimal payoff for the original market* $M(g, s)$).

PROOF: Let $\bar{\mu}$ be compatible with $(\underline{u}, \bar{v})$ in $M(g, s)$; we first show that if each q chooses $\sigma_q = \bar{v}_q$, leaving g_q unchanged, $\bar{\mu}$ is an outcome with Q-payoff $\bar{v}$. To see this, note that $(\underline{u} \wedge \underline{u}(\bar{v}), \bar{v} \vee \bar{v}(\bar{v}))$ is a stable payoff of $M(g, s)$ (see assertion (4.1), Lemma 2, where the roles of P and Q are reversed and $r^1 = s$, $r^2 = \bar{v}$). Thus $\bar{v}(\bar{v}) \leqslant \bar{v}$ since $\bar{v}(\bar{v}) \geqslant \bar{v}$. By individual rationality $\bar{v}(\bar{v}) = \bar{v}$ and there is a unique stable payoff in $M(g, \bar{v})$ which is $(\underline{u}, \bar{v})$ and in particular $(\bar{u}(\bar{v}), \underline{v}(\bar{v})) = (\bar{u}(\bar{v})$; since $\bar{\mu}$ is compatible with $(\underline{u}, \bar{v})$ it is an outcome of $\bar{v}$ and $\tilde{v}^{\bar{\mu}}(\bar{v}) = \bar{v}$, since if q is unmatched by $\bar{\mu}$, $s_q = \bar{v}_q$. We need to show now that $(\bar{v}, \bar{\mu})$ is a strong equilibrium. Suppose there is Q, $\sigma_{|Q'}$ and μ outcome of $\sigma = (\sigma_{|Q'}, \bar{v}_{|Q-Q'})$ such that $\tilde{v}^{\mu}_q(\sigma) > \bar{v}_q$ for every q in Q'. Then we have a contradiction with Theorem 2 applied to Q agents when their true utility functions are $(g_q, \bar{v}_q)$. *Q.E.D.*

We restrict our attention now to the manipulation of the reservation prices, i.e., we suppose $g_q = \gamma_q$ for every q and γ is now omitted in the notation. This is always the case in a buyer-seller market where sellers are not allowed to discriminate. Then for each possible μ, the Q-payoff is:

$$\begin{aligned} \tilde{v}^{\mu}_q(\sigma) &= \underline{v}_q(\sigma) \quad \text{if } q \text{ is matched by } \mu, \\ &= s_q \quad \text{otherwise.} \end{aligned}$$

We will prove that $\bar{v}$ is the only Nash payoff whenever agents announce σ above s and that announcing σ_q below s_q is a dominated strategy for agent q.

Since the payoffs are not uniquely determined by the strategies there are two notions a priori of Nash equilibrium: (σ, μ) is a *Nash equilibrium* in the weak (resp. strong) sense if μ is an outcome of σ and if for no q_0, there is σ' with $\sigma'_q = \sigma_q$, $q \neq q_0$ such that, for every (resp. for one) outcome μ' of σ', $\tilde{v}^{\mu'}_{q_0}(\sigma') > \tilde{v}^{\mu}_{q_0}(\sigma)$. In the weak concept, it is supposed that an agent changes his strategy when he is sure to be better off, while in the strong one he changes it whenever he has a chance to be better off. Actually the two concepts are equivalent here as is implied by the following lemma.

LEMMA 6: *Let* μ *be an outcome of* σ *with* $\tilde{v}^{\mu}(\sigma) \geqslant s$. *It is equivalent to say: there is* q *in* Q *and* (i) *there is* σ^1_q *and* μ_1 *outcome of* σ^1 *where* $\sigma^1 = (\sigma^1_q, \sigma_{\hat{q}})$, $\tilde{v}^{\mu_1}_q(\sigma^1) > \tilde{v}^{\mu}_q(\sigma)$; (ii) *there is* σ^2_q *and* μ_2 *outcome of* σ^2 *such that* $\tilde{v}^{\mu_2}_q(\sigma^2) > \tilde{v}^{\mu}_q(\sigma)$; (iii) *there is* σ^3_q *such that* $\bar{v}_q(\sigma^3) > \sigma^3_q > \tilde{v}^{\mu}_q(\sigma)$.

PROOF: (i) implies (iii): Since $\tilde{v}^{\mu_1}_q(\sigma^1) > \tilde{v}^{\mu}_q(\sigma) \geqslant s_q$, q is matched under μ_1 and $\tilde{v}^{\mu_1}_q(\sigma^1) = \underline{v}_q(\sigma^1)$. Take now σ^3 such that (iv) $\underline{v}_q(\sigma^1) > \sigma^3_q > \tilde{v}^{\mu}_q(\sigma)$; $(\bar{u}(\sigma^1), \underline{v}(\sigma^1))$ is a stable payoff in $M(\sigma^3)$ with compatible matching μ_1; indeed the only change between $M(\sigma^1)$ and $M(\sigma^3)$ is the reservation price of q but since q is matched under μ_1, we need only check the individual rationality condition which is satisfied since $\underline{v}_q(\sigma^1) > \sigma^3_q$. This implies $\bar{v}(\sigma^3) \geqslant \underline{v}(\sigma^1)$ and using (iv) we get $\bar{v}_q(\sigma^3) > \sigma^3_q > \tilde{v}^{\mu}_q(\sigma)$, which gives (iii).

To show that (iii) implies (ii) it suffices to take $\sigma_q^2 = \sigma_q^3$ since by property 1(A) $\bar{v}_q(\sigma^2) > \sigma_q^2$ implies that q is matched under every stable matching in $M(\sigma^2)$ and thus $\tilde{v}_q^{\mu_2}(\sigma^2) = \underline{v}_q(\sigma^2) \geqslant \sigma_q^2$ for every μ_2 outcome of σ^2. (ii) implies (i) is obvious by taking $\sigma_q^1 = \sigma_q^2$. *Q.E.D.*

COROLLARY 6: (A) *Every Nash equilibrium in the weak sense is a Nash equilibrium in the strong sense.* (B) *If* (σ, μ) *is a Nash equilibrium,* $\bar{v}_q(\sigma) = \underline{v}_q(\sigma)$ *for every* q *and* $\bar{v}_q(\sigma_q, s_q) = s_q$ *for every unmatched* q.

PROOF: (A) follows from the equivalence of (i) and (ii) of Lemma 6 and from the fact that $\tilde{v}^{\mu}(\sigma) \geqslant s$ is always satisfied at an equilibrium since $\sigma_q = s_q$ is a strategy which ensures s_q to q (Lemma 7 below).

To show (B) suppose on the contrary $\bar{v}_q(\sigma) > \underline{v}_q(\sigma)$ for some q. Then necessarily q is matched (property 1(A)) and let σ_q' be such that $\bar{v}_q(\sigma) > \sigma_q' > \underline{v}_q(\sigma)$. Then $\sigma_q' > \sigma_q$ and thus $\bar{v}_q(\sigma_{\hat{q}}, \sigma_q') \geqslant \bar{v}_q(\sigma) > \sigma_q' > \underline{v}_q(\sigma)$, but since q is matched, $\tilde{v}_q^{\mu}(\sigma) = \underline{v}_q(\sigma)$ and this contradicts (iii) of Lemma 6. The other assertion is proved in the same way. *Q.E.D.*

We shall consider now that an agent always uses a strategy above his reservation price as is justified by Lemma 7.

LEMMA 7: *Truth* $(\sigma_q = s_q)$ *ensures a payoff of* s_q *to agent* q *and dominates every* σ_q, $\sigma_q > s_q$.

PROOF: Let $\sigma_q = s_q$; if q is matched his payoff is $\underline{v}_q(\sigma) \geqslant \sigma_q = s_q$ and if q is unmatched his payoff is s_q. This proves the first assertion. Suppose now $\sigma_q < s_q$, let $\sigma_{\hat{q}}$ be any strategies of others players than q, and μ an outcome of σ; if $\tilde{v}_q^{\mu}(\sigma) \leqslant s_q$ by the remark above strategy s_q is at least as good as σ_q; if $\tilde{v}_q^{\mu}(\sigma) > s_q$ then surely q is matched by μ and $\tilde{v}_q^{\mu}(\sigma) = \underline{v}_q(\sigma) > s_q$; since $\underline{v}_q$ is increasing $\underline{v}_q(\sigma_{\hat{q}}, s_q) \geqslant \underline{v}_q(\sigma) > s_q$ and q by announcing s_q will surely get $\underline{v}_q(\sigma_{\hat{q}}, s_q)$ which is not worse than $\tilde{v}_q^{\mu}(\sigma)$. *Q.E.D.*

LEMMA 8: *Let* (u, v) *be a stable payoff of* $M(\sigma)$ *with compatible matching* μ; *then* (u, v') *is a stable payoff of* $M(\sigma')$ *with compatible matching* μ *where* σ' *is any* Q*-vector and*

$$v_q' = v_q, \quad \sigma_q' \leqslant \sigma_q \quad \text{if } q \text{ is matched by } \mu,$$

$$v_q' = \sigma_q', \quad \sigma_q' \geqslant \sigma_q \quad \text{otherwise.}$$

PROOF: Let x be the transfers associated to μ and yielding the payoff (u, v) in $M(\sigma)$. The payoff in $M(\sigma')$ associated to this allocation are exactly (u, v') so (3.2) and (3.3) are satisfied (feasibility); (3.4) (individual rationality) is true since if q is matched $v_q = v_q' \geqslant \sigma_q \geqslant \sigma_q'$; (3.5) (group rationality) is true since $v_q' \geqslant v_q$ is always true. *Q.E.D.*

LEMMA 9: *Let (σ, μ) be a Nash equilibrium with $\sigma \geqslant s$ and $\sigma_q = s_q$ for every q unmatched. Then $\tilde{v}^{\mu}(\sigma) = \bar{v}$.*

PROOF: From Lemma 8, the payoff $(\bar{u}(\sigma), \underline{v}(\sigma))$ is stable in $M(s)$, so $\underline{v}(\sigma) \leqslant \bar{v}$. On the other hand, $\tilde{v}^{\mu}(\sigma) = \underline{v}(\sigma) = \bar{v}(\sigma)$, since if q is matched, $\tilde{v}^{\mu}_q(\sigma) = \underline{v}_q(\sigma) = \bar{v}_q(\sigma)$ (Corollary 6), and if q is unmatched, $\tilde{v}^{\mu}_q(\sigma) = s_q$ and $\underline{v}_q(\sigma) = \bar{v}_q(\sigma) = \sigma_q = s_q$. But $\sigma \geqslant s$ implies $\bar{v}(\sigma) \geqslant \bar{v}(s) = \bar{v}$ and finally $\tilde{v}^{\mu}(\sigma)$, $\underline{v}(\sigma)$, $\bar{v}(\sigma)$ and $\bar{v}$ are all equal. *Q.E.D.*

Lemma 9 is the result we look for in the special case where unmatched agents announce their reservation price; if q is unmatched at an equilibrium and $\sigma_q > s_q$ he is surely indifferent between the strategies σ_q and s_q but by playing σ_q he might make some matched agents better off (since $\underline{v}$ is increasing in σ) without making himself worse. Actually it is not the case.

THEOREM 4: *The Q-payoff $\tilde{v}^{\mu}(\sigma)$ of a Nash equilibrium (σ, μ), where $\sigma \geqslant s$, is equal to $\bar{v}$.*

PROOF: We prove that if (σ, μ) is a Nash equilibrium, $(\sigma \geqslant s)$, where an agent q_0 is unmatched with $\sigma_{q_0} > s_{q_0}$, then $((\sigma_{\hat{q}_0}, s_{q_0}), \mu)$ is still a Nash equilibrium with the same payoff. Applying this result inductively for each unmatched agent q with $\sigma_q > s_q$, one get a Nash equilibrium with payoff $\tilde{v}^{\mu}(\sigma)$ and which satisfies the condition of Lemma 9, so $\tilde{v}^{\mu}(\sigma) = \bar{v}$.

Thus suppose q_0 unmatched, with $\sigma_{q_0} > s_{q_0}$. Note $\tilde{\sigma} = (\sigma_{q_0}, s_{q_0})$ and take σ'_{q_0}, $\sigma_{q_0} > \sigma'_{q_0} > s_{q_0}$; if q_0 may be matched in a stable matching of $M(\sigma_{\hat{q}_0}, \sigma'_{q_0})$ with payoff (u, v), the payoff (u, v) is still stable in $M(\tilde{\sigma})$ (Lemma 8) and $\bar{v}_{q_0}(\tilde{\sigma}) > s_{q_0}$ which is in contradiction with Corollary 6, so by taking a limit of stable payoffs with compatible matchings for $M(\sigma_{\hat{q}_0}, \sigma'_{q_0})$, where $\sigma'_{q_0} \to s_{q_0}$, one gets a stable payoff of $M(\tilde{\sigma})$ compatible with a matching where q_0 is not matched. By Property 1(C) q_0 may be not matched for every stable payoff and in particular for $(\bar{u}(\tilde{\sigma}), \underline{v}(\tilde{\sigma}))$; Lemma 8 then implies that the same payoff (except that $\underline{v}_{q_0}(\tilde{\sigma})$ is changed in σ_{q_0}) is stable in $M(\sigma)$ so $\underline{v}_q(\tilde{\sigma}) \geqslant \underline{v}_q(\sigma) = \bar{v}_q(\sigma)$ if $q \neq q_0$. Since $\bar{v}_q(\tilde{\sigma}) \geqslant \bar{v}_q(\sigma)$ is always true we get $\underline{v}_q(\tilde{\sigma}) = \underline{v}_q(\sigma)$ if $q \neq q_0$, $\underline{v}_{q_0}(\tilde{\sigma}) = s_{q_0}$, and $\bar{u}(\sigma) = \bar{u}(\tilde{\sigma})$. Obviously μ is compatible with $(\bar{u}(\tilde{\sigma}), \underline{v}(\tilde{\sigma}))$ in $M(\tilde{\sigma})$; thus μ is an outcome of $\tilde{\sigma}$ with payoff $\tilde{v}^{\mu}(\sigma)$.

It remains to show that $(\tilde{\sigma}, \mu)$ is a Nash equilibrium. Suppose not, there would exist, by Lemma 6, q and σ'_q such that

$$\bar{v}_q(\tilde{\sigma}') > \sigma'_q > \tilde{v}^{\mu}_q(\tilde{\sigma}) = \tilde{v}^{\mu}_q(\sigma) \quad \text{where} \quad \tilde{\sigma}' = (\tilde{\sigma}_{\hat{q}}, \sigma_{q'}).$$

Since $\bar{v}$ is increasing in σ and $\tilde{\sigma}_{\hat{q}} \leqslant \sigma_{\hat{q}}$ (since $\sigma_{q_0} > s_{q_0}$) we should have $\bar{v}_q(\sigma_{\hat{q}}, \sigma'_q) \geqslant \bar{v}_q(\sigma') > \sigma'_q > \tilde{v}^{\mu}_q(\sigma)$ and (σ, μ) would not be a Nash equilibrium (Lemma 6). *Q.E.D.*

The following example justifies the restriction $\sigma \geqslant s$ in the statement of Theorem 4. Suppose there are two sellers and one buyer with linear utility functions represented by the matrix:

0_1	0_2
2	1

The true reservation prices are $(s_1, s_2) = (1, 1)$. Then $\bar{v} = (2, 1)$, but one may check that (σ, μ) where $\sigma = (1, 0)$, and μ matches 0_1 with the buyer is a Nash equilibrium with payoff $(1, 1)$.

Laboratoire d'Econometrie de L'Ecole Polytechnique, Paris
and
University of California, Berkeley

Manuscript received October, 1983; revision received August, 1984.

REFERENCES

[1] CRAWFORD, V. P., AND E. L. KNOER: "Job Matching with Heterogeneous Firms and Workers," *Econometrica*, 49(1981), 437-450.
[2] DEMANGE, G.: "Strategyproofness in the Assignment Market Game," Laboratoire d'Econometrie de l'Ecole Polytechnique, Paris, June, 1982.
[3] DUBINS, L. E., AND D. FREEDMAN: "Machiavelli and the Gale-Shapley Algorithm," *American Mathematical Monthly*, 88(1981), 485-494.
[4] GALE, D.: *The Theory of Linear Economic Models*. New York: McGraw Hill, 1960.
[5] ———: "Equilibrium in a Discrete Exchange Economy with Money," *International Journal of Game Theory*, 13(1984), 41-60.
[6] GALE, D., AND L. S. SHAPLEY: "College Admission and the Stability of Marriage," *American Mathematical Monthly*, 69(1962), 9-14.
[7] KNUTH, D.: *Marriages Stables*. Montreal: Montreal University Press, 1976.
[8] LEONARD, H. B.: "Elicitation of Honest Preferences for the Assignment of Individuals to Positions," *Journal of Political Economy*, 91(1983), 461.
[9] QUINZII, M.: "Core and Competitive Equilibria With Indivisibilities," *International Journal of Game Theory*, 13(1984), 41-60.
[10] ROTH, A.: "The Economics of Matching: Stability and Incentives," *Mathematics of Operations Research*, 7(1982), 617-628.
[11] SHAPLEY, L. S., AND M. SHUBIK: "The Assignment Game, I: The Core," *International Journal of Game Theory*, 1(1972), 111-130.
[12] VICKREY, W.: "Counterspeculation, Auctions and Competitive Sealed Tenders," *Journal of Finance*, 15(1961), 8-37.