

A geometric approach to regularity of optimal maps

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(with Brendle, Léger, Rankin '24 and Y-H Kim, M Warren '10)

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click on 'Talk'

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Minimal hypersurfaces in \mathbf{R}^{n+1}

$$u \in \arg \min_{u|_{\partial\Omega}=f} \int_{\Omega} \sqrt{1 + |\nabla u|^2} d^n x \quad \text{'minimizing'}$$

satisfies
$$0 = \nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) \quad \text{'minimal'}$$

Blow-up: $u(0) = 0 = \nabla u(0)$ yields $u_0(x) = \lim_{r_k \rightarrow 0} r_k^{-2} u(r_k x)$ minimal on \mathbf{R}^n

THM (Bernstein '14, deGiorgi '65, Almgren '66, Simons '68): If $n < 7$ then u_0 is linear.

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COUNTEREXAMPLES (Bombieri-deGiorgi-Giusti '68) whenever $n \geq 7$.

HIGHER CODIMENSION:

- (Federer '69) each algebraic curve (or analytic variety $p(z) = 0$ in \mathbf{C}^n) is minimal
- for analogous minimization in higher codimension, singularities have codimension ≥ 2 (Almgren '00)

Maximal spacelike hypersurfaces in Minkowski space $\mathbf{R}^{n,1}$

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PROS: holds for all $n \in \mathbf{N}$

CONS:

- $SO(n, 1)$ is noncompact, unlike $SO(n + 1)$.
- uniformity of ellipticity degenerates as $|\nabla u_0| \rightarrow 1$;
- orientation delicacies (associated e.g. with disconnectedness of S^0)

What about spacelike n -volume maximizers in e.g. $\mathbf{R}^{n,m}$?

- much less is known (Mealy '91, Harvey–Lawson '12)

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THM 1 (Kim–M.–Warren '10):

graphs of optimal maps are **spacelike maximizing** (with $m = n$)

THM 2 (Brendle–Léger–M.–Rankin '24) A sign becomes favorable in the pseudo-Riemannian setting (relative to the Riemannian case) allowing us to give a new proof of Ma–Trudinger–Wang's (2005) **regularity results**.

Submanifold Geometry

Let $\Sigma^n \subset \hat{M}^{n+m}$ be a maximal spacelike submanifold of a manifold \hat{M} equipped with a signature (n, m) metric $\hat{g}(\cdot, \cdot)$ and its associated Riemann tensor $\hat{R}(\cdot, \cdot, \cdot, \cdot)$. Here *spacelike* means $g := \hat{g}|_{(T\Sigma)^2} > 0$, *maximal* means zero mean curvature vector $H = \text{tr}_M \mathbb{I} = 0$ and $\mathbb{I}_z : (T_z\Sigma)^2 \rightarrow (T_z\Sigma)^\perp$ is the *second fundamental form*

$$\mathbb{I}(X, Y) := \hat{D}_X Y - D_X Y,$$

i.e. the difference between the \hat{g} -covariant derivative \hat{D} and g -covariant derivative D on tangent fields X, Y to Σ .

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i.e. the difference between the \hat{g} -covariant derivative \hat{D} and g -covariant derivative D on tangent fields X, Y to Σ . Let e_1, \dots, e_n diagonalize S and $\hat{E}_1, \dots, \hat{E}_{n+m}$ be local \hat{g} -orthonormal frames on Σ and \hat{M} respectively.

Lemma (Brendle–Léger–M.–Rankin '24)

If \hat{S} is any positive-definite symmetric $(0,2)$ -tensor on \hat{M} and $S = \hat{S}|_{(T\Sigma)^2}$, there is a constant $c = c(\|\hat{g}, \hat{g}^{-1}, \hat{S}\|_{C^2(\{z\})})$ independent of Σ such that

$$\frac{(\Delta_g S)(e_n, e_n)}{2S(e_n, e_n)} \geq \sum_{l=1}^n (\hat{R}(e_l, e_n, e_l, e_n) - cS(e_l, e_l))$$

Proof sketch: After a long computation exploiting maximality ($H = 0$),

$$\begin{aligned} \frac{\Delta S}{2}(e_n, e_n) &= \sum_{l=1}^n \left[\frac{1}{2}(\hat{D}_{e_l, e_l}^2 \hat{S})(e_n, e_n) + 2(\hat{D}_{e_l} \hat{S})(\mathbb{I}(e_l, e_n), e_n) \right. \\ &\quad \left. + \hat{S}(\mathbb{I}(e_l, e_n), \mathbb{I}(e_l, e_n)) - \sum_{\alpha, \beta=1}^{n+m} \hat{g}^{\alpha\beta} \hat{R}(e_l, e_n, e_l, \hat{E}_\alpha) \hat{S}(\hat{E}_\beta, e_n) \right. \\ &\quad \left. + S(e_n, e_n) \left[\hat{R}(e_l, e_n, e_l, e_n) - \hat{g}(\mathbb{I}(e_l, e_n), \mathbb{I}(e_l, e_n)) \right] \right] \end{aligned}$$

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 &\quad + \hat{S}(\mathbb{I}(e_l, e_n), \mathbb{I}(e_l, e_n)) - \sum_{\alpha, \beta=1}^{n+m} \hat{g}^{\alpha\beta} \hat{R}(e_l, e_n, e_l, \hat{E}_\alpha) \hat{S}(\hat{E}_\beta, e_n) \\
 &\quad \left. + S(e_n, e_n) \left[\hat{R}(e_l, e_n, e_l, e_n) - \hat{g}(\mathbb{I}(e_l, e_n), \mathbb{I}(e_l, e_n)) \right] \right] \\
 &\geq S(e_n, e_n) \sum_{i=1}^n [\hat{R}(e_l, e_n, e_l, e_n) - cS(e_l, e_l)]
 \end{aligned}$$

□

Corollary (A priori lower bound for the maximum of $S(e_n, e_n)$)

If $(z, e_n) \in T\Sigma$ *maximize* S locally, then $0 \geq (\Delta_g S)(e_n, e_n)$ hence

$$\mu_n := S(e_n, e_n) \geq \frac{1}{c} \sum_{l=1}^n \hat{R}(e_l, e_n, e_l, e_n) = \frac{1}{c} \text{tr}_\Sigma \hat{R}(\cdot, e_n, \cdot, e_n).$$

Optimal transport

$b(x, y)$ 'benefit' per unit mass transported from $x \in \Omega$ to $\bar{x} \in \bar{\Omega}$
 $\Omega, \bar{\Omega} \subset \subset \mathbf{R}^n$ open and bounded (or oriented manifolds); 'landscapes';
n-forms $0 < \mu, \bar{\mu}$ on $\Omega, \bar{\Omega}$; normalized densities of supply and demand
 $\mu(x) = \rho(x) dx^1 \wedge \dots \wedge dx^n$ and $\bar{\mu}(\bar{x}) = \bar{\rho}(\bar{x}) d\bar{x}^1 \wedge \dots \wedge d\bar{x}^n$

MONGE (1781): seek

$$\sup_{F_{\#}\mu = \bar{\mu}} \int_{\Omega} b(x, F(x)) \mu$$

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MONGE (1781): seek

$$\sup_{F_{\#}\mu = \bar{\mu}} \int_{\Omega} b(x, F(x)) \mu = \min_{b \leq u \oplus \bar{u}} \int_{\Omega} u \mu + \int_{\bar{\Omega}} \bar{u} \bar{\mu}$$

KANTOROVICH (1942)

- $\det DF(x) = \pm \rho(x) / \bar{\rho}(F(x))$ if $F : \Omega \rightarrow \bar{\Omega}$ is a diffeomorphism

HYPOTHESES (Ma-Trudinger-Wang '05)

(A0) $b \in C^4(\text{cl}(\Omega \times \bar{\Omega}))$ and for each $x \in \text{cl}(\Omega)$:

(A1) $\bar{x} \in \text{cl}(\bar{\Omega}) \mapsto D_x b(x, \bar{x}) := \left(\frac{\partial b}{\partial x^1}, \dots, \frac{\partial b}{\partial x^n} \right)$ is a **diffeomorphism**;

(A2) with **convex** range $\bar{\Omega}_x = D_x b(x, \text{cl}(\bar{\Omega}))$

DEFN: $t \in [0, 1] \mapsto (x, \bar{x}_t) \in \text{cl}(\Omega \times \bar{\Omega})$ is called a **b -segment** if

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$$\frac{d^2}{dt^2} [D_x b(x, \bar{x}_t)] = 0 \quad \forall t \in [0, 1]$$

Assume $b^*(\bar{x}, x) := b(x, \bar{x})$ also satisfies (A0)-(A2) and

$$(A3) \quad \left. \frac{\partial^2 b}{\partial s \partial t} \right|_{s=0=t} (x_s, \bar{x}_t) = 0 \quad \implies \quad \left. \frac{\partial^4 b}{\partial^2 s \partial^2 t} \right|_{s=0=t} (x_s, \bar{x}_t) > 0$$

whenever (x_0, \bar{x}_t) is a b -segment (and $(x_s)_{s \in [0,1]} \in C^2$)

Theorem (Gangbo '95, Levin '96; Gangbo–McCann '95–'96)

If (A0–A1) a unique minimizer $F_{\#}\mu = \bar{\mu}$ exists.

Theorem (Ma–Trudinger–Wang '05; interior regularity)

If also (A2–A3) and $\log \rho, \log \bar{\rho} \in C^{k,\alpha}$, for $k \geq 2$ and $0 < \alpha < 1$, then $F \in C_{loc}^{k+1,\alpha}(\Omega, \bar{\Omega})$.

- first regularity result for an open class of costs $c = -b$
- subsequent improvements / related results by many authors
- Loeper '10: if $(\overline{A3})$ fails $\exists \log \rho, \log \bar{\rho} \in C^\infty$ for which F discontinuous

A geometric view (Kim–M. '10)

RMK: Kantorovich $\gamma = (id \times F)_{\#}\mu$ satisfies $\Delta \geq 0$ on $\Sigma \times \Sigma := (\text{spt}\gamma)^2$,
where

$$\begin{aligned}\Delta(x, \bar{x}; x_0, \bar{x}_0) &= b(x, \bar{x}) + b(x_0, \bar{x}_0) - b(x, \bar{x}_0) - b(x_0, \bar{x}) \\ &=: \Delta_0(x, \bar{x}).\end{aligned}$$

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Fix $(x_0, \bar{x}_0) \in \hat{M} := \Omega \times \bar{\Omega}$. Taylor expanding $\Delta_0(x, \bar{x})$ around (x_0, \bar{x}_0) yields

$$\Delta_0(x_0 + \delta x, \bar{x}_0 + \delta \bar{x}) =$$

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$$\begin{aligned}\Delta_0(x_0 + \delta x, \bar{x}_0 + \delta \bar{x}) &= \frac{1}{2}(\delta x, \delta \bar{x}) \text{Hess } \Delta_0 \begin{pmatrix} \delta x \\ \delta \bar{x} \end{pmatrix} + O(|\delta x| + |\delta \bar{x}|)^3 \\ &= \sum_{i,j=1}^n \delta x^i \delta \bar{x}^j \frac{\partial^2 b}{\partial x^i \partial \bar{x}^j} + O(|\delta x| + |\delta \bar{x}|)^3\end{aligned}$$

- $\hat{h} := \text{Hess}_{(x_0, \bar{x}_0)} \Delta_0$ is pseudo-Riemannian since $\det \frac{\partial^2 b}{\partial x^i \partial \bar{x}^j} \neq 0$ by (A1)
- its signature is (n, n) since $(\delta x, \pm \delta \bar{x})$ flips the sign of the sum above
- $\Sigma := \text{spt}\gamma$ is *rectifiably nontimelike*, since $h = \hat{h}|_{T\Sigma^2} \geq 0$ by RMK above.

- e.g. for $b(x, y) = x \cdot y$,

$$\begin{aligned}\Delta_0(x, y) &:= b(x, y) + b(x_0, y_0) - b(x, y_0) - b(x_0, y) \\ &= (x - x_0) \cdot (y - y_0)\end{aligned}$$

and

$$\text{Hess}_{(x_0, y_0)} \Delta_0 = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}$$

- more generally,

$$\hat{h} := \text{Hess}_{(x_0, y_0)} \Delta_0 = \begin{bmatrix} 0 & D_{x^i y^j}^2 b(x_0, y_0) \\ D_{x^i y^j}^2 b(x_0, y_0)^T & 0 \end{bmatrix}$$

so

$$\Delta_0(x_0 + \delta x, y_0 + \delta y) = -\Delta_0(x_0 + \delta x, y_0 - \delta y) + \text{l.o.t.}$$

- thus \hat{h} has signature (n, n) , depends only on b
- (Kim-M. '10) (A2) \Leftrightarrow geodesic convexity of each $\{x\} \times \bar{\Omega}$ in $(\Omega \times \bar{\Omega}, \hat{h})$
- note $\{x\} \times \bar{\Omega}$ and similarly $\Omega \times \{\bar{x}\}$ are both \hat{h} -null

Conformal and calibrated geometries

THM (Kim–M. '10) If (A0)–(A2) then (A3) \Leftrightarrow

$\hat{R}(p \oplus 0, 0 \oplus \bar{p}, p \oplus 0, 0 \oplus \bar{p}) > 0$ whenever $\hat{h}(p \oplus 0, 0 \oplus \bar{p}) = 0$.

Theorem (Kim–M.–Warren '10 spacelike maximizing)

b-optimality of γ implies $\Sigma = \text{spt}(\gamma)$ is volume maximizing (wrt compactly supported perturbations) for a conformally equivalent metric $\hat{g} = \chi \hat{h}$, with conformal factor $\chi(x, \bar{x}) > 0$ chosen so that the volume form $\text{vol}_{\hat{g}} = \pm \mu \wedge \bar{\mu}$, (i.e. has Lebesgue density $\rho(x)\bar{\rho}(\bar{x})$ on $\hat{M} = \Omega \times \bar{\Omega}$).

- In particular Σ has zero mean curvature wrt the metric \hat{g} .
- above characterizations of (A2) and (A3) also work with \hat{g} in place of \hat{h} .

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Proof sketch: $\Phi = \frac{1}{2}(\mu + \bar{\mu})$ is a **calibration** of Σ ; i.e. $d\Phi = 0$ and

$\Phi_z(\wedge_{i=1}^n v_i) \geq \|\wedge_{i=1}^n v_i\|_{\hat{g}}$ on the n -Grassmannian⁺ of \hat{M} with equality a.e. on $\wedge^n T_z \Sigma$, so for $\Sigma - \Sigma' = \partial \Lambda$

$$\text{vol}_g \Sigma = \int_{\Sigma} \Phi = \int_{\Sigma'} \Phi \geq \text{vol}_g \Sigma'.$$



Fix $(s_{ij}) > 0$ (say Euclidean) on Ω . Then the induced Riemannian metric

$$\hat{S} := \sum_{i,j=1}^n \left(s_{ij} dx^i \otimes dx^j + \chi^2 \sum_{k,l=1}^n s^{ij} \frac{\partial^2 b}{\partial x^i \partial \bar{x}^k} \frac{\partial^2 b}{\partial x^j \partial \bar{x}^l} d\bar{x}^k \otimes d\bar{x}^l \right)$$

satisfies $\text{vol}_{\hat{S}} = \mu \wedge \bar{\mu}$ on $\hat{M} = \Omega \times \bar{\Omega}$. (A0–A3) yields $\kappa > 0$ such that

$$\hat{R}_{\hat{S}}(p \oplus 0, 0 \oplus \bar{p}, p \oplus 0, 0 \oplus \bar{p}) \geq \kappa |p \wedge \bar{p}|_{\hat{S}}^2 \quad \forall \text{null } (z, p \oplus \bar{p}) \in T\hat{M}.$$

Theorem (Brendle–Léger–M.–Rankin '24 a priori spacelike estimate)

If $0 \leq \hat{\phi} \in C_c^\infty(\Omega \times \bar{\Omega})$ and $F_{\#}\mu = \bar{\mu}$ is a smooth b -optimal diffeomorphism then

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If $0 \leq \hat{\phi} \in C_c^\infty(\Omega \times \bar{\Omega})$ and $F_{\#}\mu = \bar{\mu}$ is a smooth b -optimal diffeomorphism then

$$0 < (\kappa \hat{\phi}^2)^{n-1} S \leq cg$$

on $\Sigma = \text{Graph}(F) \subset \Omega \times \bar{\Omega} = \hat{M}$, where $(\phi, S, g) = (\hat{\phi}, \hat{S}, \hat{g})|_{\Sigma}$ and $c = c(\|\hat{g}, \hat{g}^{-1}, \hat{S}, \hat{\phi}\|_{C^2(\text{spt } \hat{\phi})}, \|\log \frac{\mu}{\text{vol}_S}\|_{C^0})$ is independent of $\mu, \bar{\mu}$.

- after this regularity follows by local replacement using continuity method

Proof sketch: Kantorovich dual potentials satisfy

$$u(x) + \bar{u}(\bar{x}) - b(x, \bar{x}) \geq 0$$

on \hat{M} with equality on $\Sigma = \text{Graph}(F)$. Thus

$$\begin{aligned} Du(x) - D_x b(x, F(x)) &= 0 && \text{(FOC)} \\ D^2 u(x) - D_{xx}^2 b(x, F(x)) &\geq 0. && \text{(SOC)} \end{aligned}$$

Differentiating (FOC) yields

$$D^2 u - D_{xx}^2 b(x, F(x)) = D_{x\bar{x}}^2 b(x, F(x)) DF(x)$$

whose determinant

$$\log \det[D^2 u - D_{xx}^2 b(x, F(x))] = \log \left| \frac{\rho}{\bar{\rho}} \det D_{x\bar{x}}^2 b \right|_{\bar{x}=F(x)} \in L^\infty$$

is bounded by the asserted constants. At least (SOC) becomes strict.

- If **uniform**, the **PDE** is **uniformly elliptic** and Schauder theory applies.

If $z = (x, F(x))$ maximizes the largest eigenvalue of $\phi^{2(n-1)}S$ relative to g , we can extend the Euclidean coordinates (x^1, \dots, x^n) which diagonalize $\Lambda := (D^2u - D_{xx}^2 b(x_0, F(x_0)))\chi > 0$ to Riemannian normal coordinates for \hat{S} . Taking $p_i = \frac{\partial}{\partial x^i}$ to be the eigenvector of Λ with eigenvalue λ_i , we can build a g -orthonormal basis $e_i = \frac{1}{\sqrt{2}}(\frac{p_i}{\sqrt{\lambda_i}} \oplus \sqrt{\lambda_i}\bar{p}_i)$ for $T_z\Sigma$ where

$$\bar{p}_i = \lambda_i^{-1} \sum_{k=1}^n \frac{\partial F^k}{\partial x^i} \frac{\partial}{\partial \bar{x}^k}.$$

Moreover $S(e_i, e_j) = \mu_i \delta_{ij}$ with $\mu_i = \frac{\lambda_i + \lambda_i^{-1}}{2}$. Ordering the eigenvalues so $\mu_i \leq \mu_n$, multilinearity and the special structure of the Riemann tensor $\hat{R}_{\hat{g}}$ yield

If $z = (x, F(x))$ maximizes the largest eigenvalue of $\phi^{2(n-1)}S$ relative to g , we can extend the Euclidean coordinates (x^1, \dots, x^n) which diagonalize $\Lambda := (D^2u - D_{xx}^2 b(x_0, F(x_0)))\chi > 0$ to Riemannian normal coordinates for \hat{S} . Taking $p_i = \frac{\partial}{\partial x^i}$ to be the eigenvector of Λ with eigenvalue λ_i , we can build a g -orthonormal basis $e_i = \frac{1}{\sqrt{2}}\left(\frac{p_i}{\sqrt{\lambda_i}} \oplus \sqrt{\lambda_i}\bar{p}_i\right)$ for $T_z\Sigma$ where

$$\bar{p}_i = \lambda_i^{-1} \sum_{k=1}^n \frac{\partial F^k}{\partial x^i} \frac{\partial}{\partial \bar{x}^k}.$$

Moreover $S(e_i, e_j) = \mu_i \delta_{ij}$ with $\mu_i = \frac{\lambda_i + \lambda_i^{-1}}{2}$. Ordering the eigenvalues so $\mu_i \leq \mu_n$, multilinearity and the special structure of the Riemann tensor $\hat{R}_{\hat{g}}$ yield

$$\hat{R}(e_i, e_n, e_i, e_n) \geq \frac{\kappa}{4} \left(\frac{\lambda_n}{\lambda_i} + \frac{\lambda_i}{\lambda_n} \right) - c(\mu_i + \mu_n + 1).$$

The arithmetic-geometric mean \neq and determinant $\prod_{i=1}^n \lambda_i$ bounds give

$$\sum_{i=1}^{n-1} \hat{R}_{\hat{g}}(e_i, e_n, e_i, e_n) \geq \frac{\kappa}{c} \mu_n^{\frac{n}{n-1}} - c\mu_n.$$

But our Corollary bounds this sum $\leq c\phi^{-2}\mu_n$, hence μ_n is bounded! \square

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Thank you!