Duality, Free Boundaries, Optimal Nonlinear Pricing

Robert J McCann

University of Toronto

www.math.toronto.edu/mccann ‘Talk 1’

with Kelvin Shuangjian Zhang (Fudan U) and Cale Rankin (U Toronto)

arXiv:2301.07660 and arXiv:2303.04937

23 November 2023

Outline

Monopolist's problem

Given compact sets $X \subset \mathbf{R}^m$, $Y \subset \mathbf{R}^n$, and 'direct utility'
$b(x, y)$ = value of product $y \in Y$ to buyer $x \in X$
$c(y)$ = monopolist's cost to produce $y \in Y$
$d\mu(x)$ = relative frequency of buyer $x \in X$ (as compared to $x' \in X$)

Monopolist's problem: choose price menu $v : Y \longrightarrow \mathbf{R}$ to maximize profits

Monopolist's problem

Given compact sets $X \subset \mathbf{R}^m$, $Y \subset \mathbf{R}^n$, and 'direct utility'
$b(x, y)$ = value of product $y \in Y$ to buyer $x \in X$
$c(y)$ = monopolist's cost to produce $y \in Y$
$d\mu(x)$ = relative frequency of buyer $x \in X$ (as compared to $x' \in X$)

Monopolist's problem: choose price menu $v : Y \longrightarrow \mathbf{R}$ to maximize profits

$$\tilde{\Pi}(v) := \int_X [v(y_v(x)) - c(y_v(x))] d\mu(x), \qquad \text{where}$$

Agent x's problem: choose $y_v(x)$ to maximize

$$y_v(x) \in \arg\max_{y \in Y} b(x, y) - v(y)$$

Constrain: v lower semicontinuous, $0 \in Y$ and $v(0) = 0 = c(0) = b(x, 0)$.

Examples

- airline ticket pricing
- insurance
- educational signaling
- optimal taxation: replace profit maximization with a budget constraint for providing services

Some history:

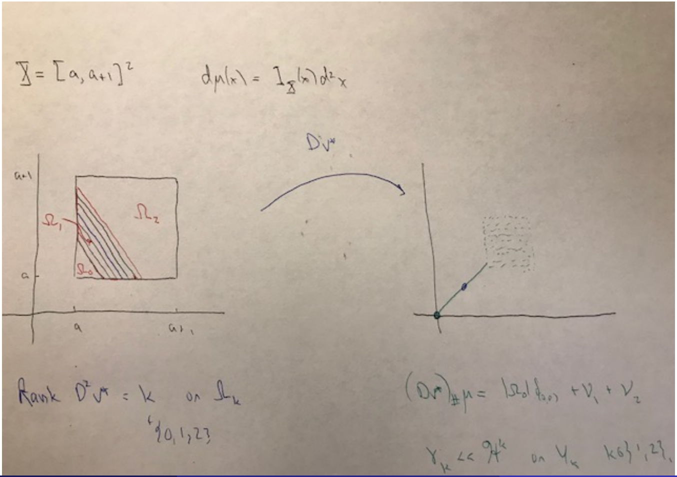
Mirrlees '71, Spence '73 ($n = 1 = m$): $\frac{\partial^2 b}{\partial x \partial y} > 0$ implies $\frac{dy_v}{dx} \geq 0$

Rochet-Choné '98 ($n = m > 1$): $b(x, y) = x \cdot y$ *bilinear* implies $y_v(x) = Dv^*(x)$ convex gradient; bunching

Some history:

Mirrlees '71, Spence '73 ($n = 1 = m$): $\frac{\partial^2 b}{\partial x \partial y} > 0$ implies $\frac{dy_v}{dx} \geq 0$

Rochet-Choné '98 ($n = m > 1$): $b(x, y) = x \cdot y$ *bilinear* implies $y_v(x) = Dv^*(x)$ convex gradient; bunching for $c(y) = \frac{1}{2}|y|^2$



Basov '03 control theoretic approach; Carlier-Lachand-Robert '03: $v^* \in C^1(\mathrm{spt}\,\mu)$; Caffarelli-Lions '06+ (unpublished) $v^* \in C^{1,1}_{loc}(X^0)$

Carlier '01: $b(x, y)$ general implies existence of optimizer $v = (v^b)^{\tilde{b}}$

Chen '13: $u \in C^1(X)$ under Ma-Trudinger-Wang (MTW) conditions, with

$$u(x) = v^b(x) := \max_{y \in Y} b(x, y) - v(y)$$

Basov '03 control theoretic approach; Carlier-Lachand-Robert '03: $v^* \in C^1(\mathrm{spt}\,\mu)$; Caffarelli-Lions '06+ (unpublished) $v^* \in C^{1,1}_{loc}(X^0)$

Carlier '01: $b(x,y)$ general implies existence of optimizer $v = (v^b)^{\tilde{b}}$

Chen '13: $u \in C^1(X)$ under Ma-Trudinger-Wang (MTW) conditions, with

$$u(x) = v^b(x) := \max_{y \in Y} b(x,y) - v(y)$$

called the 'indirect utility' to shopper x (and $\tilde{b}(y,x) = b(x,y)$)

Figalli-Kim-M. '11:
convexity of principal's problem under strengthening of (MTW) on $b(x,y)$

M.-Rankin-Zhang '23+:
$c^b \in C^2$, $\log \frac{d\mu}{dx} \in C^1 \implies u = v^b \in C^{1,1}_{loc}(X^0)$ under same strengthening

Noldeke-Samuelson (ECTA '18), Zhang (ET '19) M.-Zhang (CPAM '19):
generalize to preferences $G(x,y,z) \neq b(x,y) - z$ and profits $\pi(x,y,z) \neq z - c(y)$ nonlinear in price $z \in \mathbf{R}$

Hypotheses (Ma-Trudinger-Wang type conditions)

(B0) $b \in C^4(X \times Y)$, $m = n$, and for each $x, x_0 \in X \subset \mathbf{R}^m$:

(B1) $y \in Y \mapsto D_x b(x, y) := (\frac{\partial b}{\partial x_1}, \ldots, \frac{\partial b}{\partial x_m})$ is a diffeomorphism

(B2) with convex range $Y_x := D_x b(x, Y)$ and inverse $\bar{y}_b(\cdot, x) : Y_x \longrightarrow Y$.

Hypotheses (Ma-Trudinger-Wang type conditions)

(B0) $b \in C^4(X \times Y)$, $m = n$, and for each $x, x_0 \in X \subset \mathbf{R}^m$:

(B1) $y \in Y \mapsto D_x b(x, y) := (\frac{\partial b}{\partial x_1}, \ldots, \frac{\partial b}{\partial x_m})$ is a diffeomorphism

(B2) with convex range $Y_x := D_x b(x, Y)$ and inverse $\bar{y}_b(\cdot, x) : Y_x \longrightarrow Y$.

DEFN: $t \in [0, 1] \mapsto (x, y_t) \in X \times Y$ is called a b-segment if

$$\frac{d^2}{dt^2}[D_x b(x, y_t)] = 0 \qquad \forall t \in [0, 1]$$

(B3) Assume $t \mapsto b(x_0, y_t) - b(x, y_t)$ is convex along each b-segment

Hypotheses (Ma-Trudinger-Wang type conditions)

(B0) $b \in C^4(X \times Y)$, $m = n$, and for each $x, x_0 \in X \subset \mathbf{R}^m$:

(B1) $y \in Y \mapsto D_x b(x,y) := (\frac{\partial b}{\partial x_1}, \ldots, \frac{\partial b}{\partial x_m})$ is a diffeomorphism

(B2) with convex range $Y_x := D_x b(x, Y)$ and inverse $\bar{y}_b(\cdot, x) : Y_x \longrightarrow Y$.

DEFN: $t \in [0,1] \mapsto (x, y_t) \in X \times Y$ is called a b-segment if

$$\frac{d^2}{dt^2}[D_x b(x, y_t)] = 0 \qquad \forall t \in [0,1]$$

(B3) Assume $t \mapsto b(x_0, y_t) - b(x, y_t)$ is convex along each b-segment

- Note: when $b(x,y) = x \cdot y$ the above function of t is affine linear

Monopolists problem in terms of buyers' indirect utilities u

$$u(x) := v^b(x) := \max_{y \in Y} b(x,y) - v(y) =: b(x, y_v(x)) - v(y_v(x)) \quad (1)$$

implies

$$Du(x) = D_x b(x, y_v(x))$$

so we identify

$$y_v(x)$$

Monopolists problem in terms of buyers' indirect utilities u

$$u(x) := v^b(x) := \max_{y \in Y} b(x,y) - v(y) =: b(x, y_v(x)) - v(y_v(x)) \qquad (1)$$

implies

$$Du(x) = D_x b(x, y_v(x))$$

so we identify

$$y_v(x) = D_x b(x, \cdot)^{-1}(Du(x)) =: \bar{y}_b(Du(x), x)$$

and maximize

$$\tilde{\Pi}(v) \quad = \quad \int_X [v - c]_{\bar{y}_b(Du(x),x)} d\mu(x)$$

Monopolists problem in terms of buyers' indirect utilities u

$$u(x) := v^b(x) := \max_{y\in Y} b(x,y) - v(y) =: b(x, y_v(x)) - v(y_v(x)) \quad (1)$$

implies

$$Du(x) = D_x b(x, y_v(x))$$

so we identify

$$y_v(x) = D_x b(x,\cdot)^{-1}(Du(x)) =: \bar{y}_b(Du(x), x)$$

and maximize

$$\begin{aligned}\tilde{\Pi}(v) &= \int_X [v - c]_{\bar{y}_b(Du(x),x)} d\mu(x) \\ &= \int_X [b(x,y) - u(x) - c(y)]_{y=\bar{y}_b(Du(x),x)} d\mu(x) =: \Pi(u)\end{aligned}$$

among u of form (1) (i.e. among so called b-convex $u(\cdot) \geq 0$)

More Hypotheses and Notation

$$\max_{0 \leq u \in \mathcal{U}} \Pi(u)$$

where $\mathcal{U}$ is the set of b-convex functions:

$$\mathcal{U} := \{u \mid u(\cdot) = \sup_{y \in Y} b(\cdot, y) - v(y) \text{ on } X \text{ for some } v : Y \longrightarrow \mathbf{R} \cup \{+\infty\}\}$$

More Hypotheses and Notation

$$\max_{0\le u\in\mathcal{U}} \Pi(u)$$

where $\mathcal{U}$ is the set of b-convex functions:

$$\mathcal{U}:=\{u \mid u(\cdot) = \sup_{y\in Y} b(\cdot, y) - v(y) \text{ on } X \text{ for some } v : Y \longrightarrow \mathbf{R}\cup\{+\infty\}\}$$

(C0) $c \in C^0(Y)$
(C1) $c = (c^b)^{\tilde{b}}$ is $\tilde{b}$-convex, where $\tilde{b}(y, x) = b(x, y)$
(C2) $c^b \in C^2(X)$ (i.e. 2-uniform $\tilde{b}$-convexity of $c = c^{b\tilde{b}}$).

Carlier '01: Given (B0-B1, C0) the maximum above is attained. If $\mu << \mathcal{L}^m$ the map $x \to \bar{y}_b(Du(x), x)$ gives the consumer to product correspondence.

Figalli-Kim-M '11:

(a) If (B0-B2) hold then $\mathcal{U}$ is convex if and only if (B3) holds

Figalli-Kim-M '11:

(a) If (B0-B2) hold then $\mathcal{U}$ is convex if and only if (B3) holds

(b): If (B0-B3, C0-C2) hold then $\Pi(u)$ is 2-uniformly concave on $\mathcal{U} \subset W^{1,2}(X, d\mu)$.

If, in addition $\mu << \mathcal{L}^m$ the principal's optimal strategy u is unique μ-a.e. and stable:

i.e. $(b_i, c_i, \mu_i) \to (b_\infty, c_\infty, \mu_\infty)$ in $C^2 \times C^0 \times (C^0)^*$ implies $u_i \to u_\infty$ in $L^\infty(d\mu_\infty)$

Figalli-Kim-M '11:

(a) If (B0-B2) hold then $\mathcal{U}$ is convex if and only if (B3) holds

(b): If (B0-B3, C0-C2) hold then $\Pi(u)$ is 2-uniformly concave on $\mathcal{U} \subset W^{1,2}(X, d\mu)$.

If, in addition $\mu << \mathcal{L}^m$ the principal's optimal strategy u is unique μ-a.e. and stable:

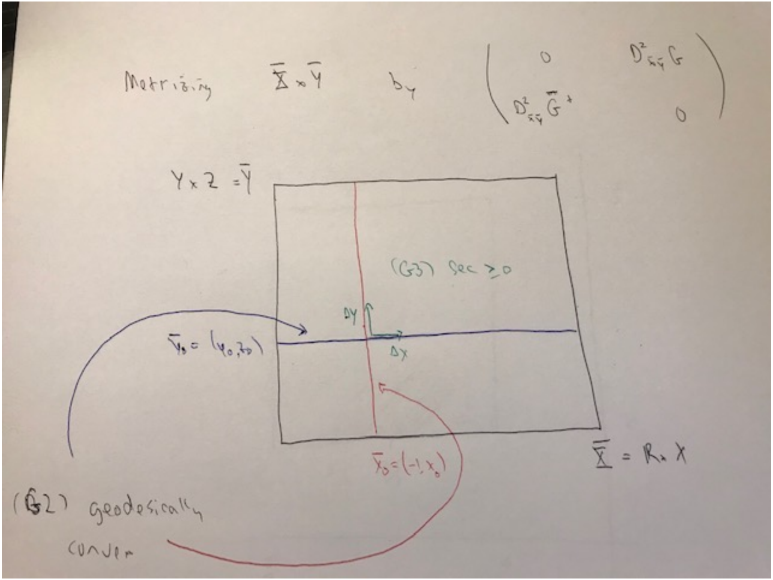
i.e. $(b_i, c_i, \mu_i) \to (b_\infty, c_\infty, \mu_\infty)$ in $C^2 \times C^0 \times (C^0)^*$ implies $u_i \to u_\infty$ in $L^\infty(d\mu_\infty)$

• the Rochet-Choné $b(x, y) = x \cdot y$ lies on the boundary of the set of preferences satisfying (B3)

• if $\|A\|_{C^1} \leq 1, \|B\|_{C^1} \leq 1$ with A convex, $b(x, y) = x \cdot y - A(x)B(y)$ satisfies (B3) if and only if B is convex

• Kim-M. '10: curvature interpretation of (B3) when $\tilde{b}(y, x) = b(x, y)$ also satisfies (B0-B2)

Pseudo-Riemannian geometry à la Kim-McCann '10



A new duality for bilinear preferences

Following Rochet-Choné '98 choose $b(x, y) = x \cdot y$ and $X, Y \subset \mathbf{R}^n$ convex so

$$\Pi(u) = \int_X [x \cdot Du - u(x) - c(Du(x))] d\mu(x)$$

with

$$u(x) = v^*(x) := \sup_{y \in Y} x \cdot y - v(y)$$

$$\in \mathcal{U} := \{u : X \longrightarrow [0, \infty] \ convex \mid Du(X) \subset Y\}$$

THM (M.-Zhang 23+
c.f. Kolesnikov-Sandomirskiy-Tsyvinski-Zimin 22+ on Beckmann auctions):

$$\max_{u \in \mathcal{U}} \Pi(u) =$$

A new duality for bilinear preferences

Following Rochet-Choné '98 choose $b(x, y) = x \cdot y$ and $X, Y \subset \mathbf{R}^n$ convex so

$$\Pi(u) = \int_X [x \cdot Du - u(x) - c(Du(x))] d\mu(x)$$

with

$$u(x) = v^*(x) := \sup_{y \in Y} x \cdot y - v(y)$$

$$\in \mathcal{U} := \{u : X \longrightarrow [0, \infty] \textit{ convex} \mid Du(X) \subset Y\}$$

THM (M.-Zhang 23+
c.f. Kolesnikov-Sandomirskiy-Tsyvinski-Zimin 22+ on Beckmann auctions):

$$\max_{u \in \mathcal{U}} \Pi(u) = \min_{S \in \mathcal{S}} \int c^*(S(x)) d\mu(x)$$

where

$$\mathcal{S} := \bigcap_{u \in \mathcal{U}} \left\{ S : X \longrightarrow \mathbf{R}^n \mid \int_X [(x - S(x)) \cdot Du - u(x)] d\mu(x) \leq 0 \right\}$$

Theorem

$$\max_{u \in \mathcal{U}} \Pi(u) = \min_{S \in \mathcal{S}} \int c^*(S(x)) d\mu(x)$$

where

$$\mathcal{S} := \bigcap_{u \in \mathcal{U}} \{S : X \longrightarrow \mathbf{R}^n \mid \langle x \cdot Du(x) - u(x) \rangle_\mu \leq \langle S(x) \cdot Du(x) \rangle_\mu\}$$

Proof: $(\leq)$: $S \in \mathcal{S}$, $u \in \mathcal{U}$ and the definition of c^* imply

$$\Pi(u) = \langle x \cdot Du(x) - u - c(Du(x)) \rangle_\mu \leq \cdots \leq \langle c^* \circ S \rangle_\mu$$

Theorem

$$\max_{u \in \mathcal{U}} \Pi(u) = \min_{S \in \mathcal{S}} \int c^*(S(x)) d\mu(x)$$

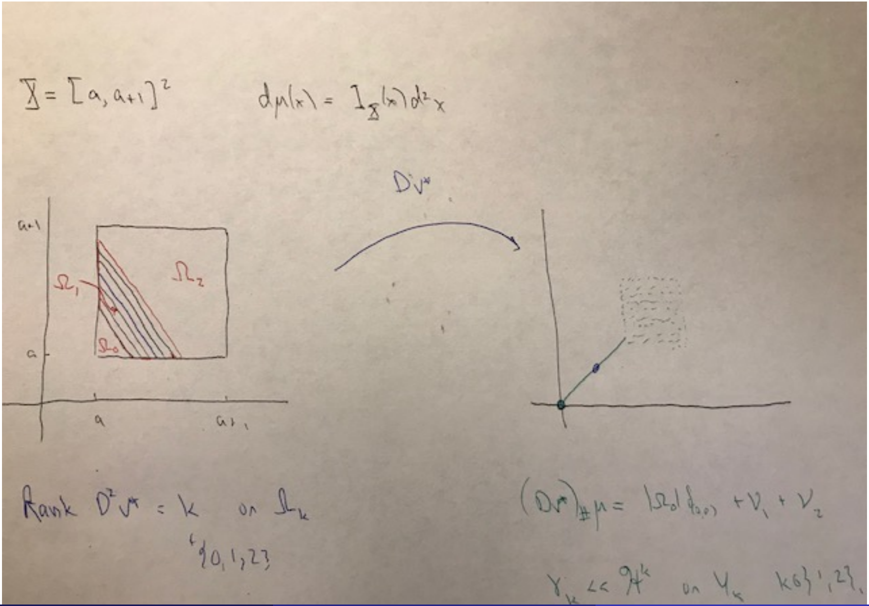
where

$$\mathcal{S} := \bigcap_{u \in \mathcal{U}} \{S : X \longrightarrow \mathbf{R}^n \mid \langle x \cdot Du(x) - u(x) \rangle_\mu \leq \langle S(x) \cdot Du(x) \rangle_\mu \}$$

Proof: $(\leq)$: $S \in \mathcal{S}$, $u \in \mathcal{U}$ and the definition of c^* imply

$$\Pi(u) = \langle x \cdot Du(x) - u - c(Du(x)) \rangle_\mu \leq \cdots \leq \langle c^* \circ S \rangle_\mu$$

$(\geq)$ Rockafellar-Fenchel duality □


X = [a, a+1]^2
dμ(x) = 1_X(x) d^2x
Dv*
a+1
a
Ω_1
Ω_2
Ω_0
a
a+1
Rank D^2 v* = k on Ω_k
∈{0,1,2}
(Dv*)_# μ = |Ω_0| δ_(0,0) + ν_1 + ν_2
ν_k << H^k on Y_k k∈{1,2}

Variational calculus for obstacle problem plus convexity

$$u \in \underset{\text{convex } u \geq 0}{\arg\max} - \int_{X:=[a,a+1]^2} \left(\frac{1}{2} |Du(x) - x|^2 + u - \frac{1}{2}|x|^2 \right) d\mu(x)$$

gives $u = u_i$ on $\Omega_i = \{x \mid \mathrm{Rank}(D^2 u(x)) = i\}$ where

- on Ω_0 exclusion: $u_0 = 0$
- on Ω_1, Euler-Lagrange ODE: if $2u_1(x_1, x_2) =: k(x_1 + x_2)$ then
$$k(s) = \tfrac{3}{4}s^2 - as - \log|s - 2a| + const$$

subject to boundary conditions $u_1 = u_0$ and $Du_1 = Du_0$ at lower boundary.

Variational calculus for obstacle problem plus convexity

$$u \in \underset{\text{convex } u \geq 0}{\arg\max} -\int_{X:=[a,a+1]^2} \left(\frac{1}{2}|Du(x) - x|^2 + u - \frac{1}{2}|x|^2\right) d\mu(x)$$

gives $u = u_i$ on $\Omega_i = \{x \mid \mathrm{Rank}(D^2 u(x)) = i\}$ where

- on Ω_0 exclusion: $u_0 = 0$
- on Ω_1, Euler-Lagrange ODE: if $2u_1(x_1, x_2) =: k(x_1 + x_2)$ then
$$k(s) = \frac{3}{4}s^2 - as - \log|s - 2a| + const$$
subject to boundary conditions $u_1 = u_0$ and $Du_1 = Du_0$ at lower boundary.
- on Ω_2 Euler-Lagrange PDE: $\Delta u_2 = 3$ subject to boundary conditions

$$(Du_2(x) - x) \cdot \hat{n}_{\Omega_2}(x) = 0 \quad \text{on} \quad \partial X \cap \bar{\Omega}_2$$
$$(Du_2 - Du_1) \cdot \hat{n}_{\Omega_2}(x) = 0 \quad \text{on} \quad \partial\Omega_2 \cap \partial\Omega_1 \quad \text{(Neumann)}$$

Variational calculus for obstacle problem plus convexity

$$u \in \underset{\text{convex } u \geq 0}{\arg\max} - \int_{X:=[a,a+1]^2} \left(\frac{1}{2}|Du(x) - x|^2 + u - \frac{1}{2}|x|^2 \right) d\mu(x)$$

gives $u = u_i$ on $\Omega_i = \{x \mid \mathrm{Rank}(D^2 u(x)) = i\}$ where

- on Ω_0 exclusion: $u_0 = 0$
- on Ω_1, Euler-Lagrange ODE: if $2u_1(x_1, x_2) =: k(x_1 + x_2)$ then

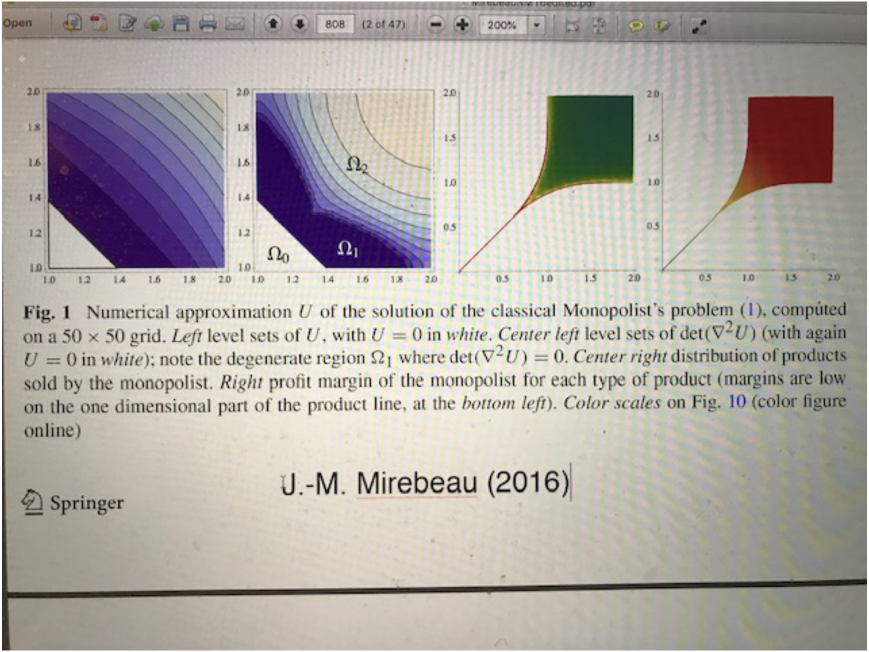
$$k(s) = \frac{3}{4}s^2 - as - \log|s - 2a| + const$$

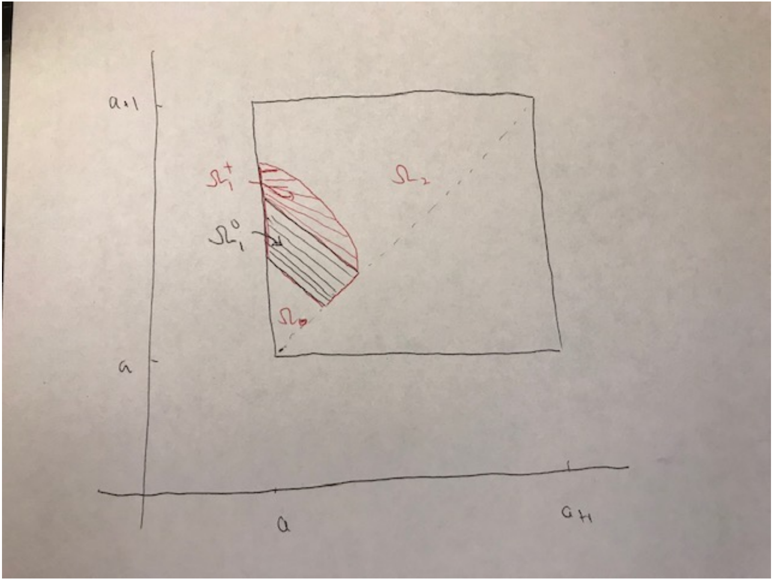
subject to boundary conditions $u_1 = u_0$ and $Du_1 = Du_0$ at lower boundary.

- on Ω_2 Euler-Lagrange PDE: $\Delta u_2 = 3$ subject to boundary conditions

$$\begin{aligned} (Du_2(x) - x) \cdot \hat{n}_{\Omega_2}(x) = 0 \quad &\text{on} \quad \partial X \cap \bar{\Omega}_2 \\ (Du_2 - Du_1) \cdot \hat{n}_{\Omega_2}(x) = 0 \quad &\text{on} \quad \partial\Omega_2 \cap \partial\Omega_1 \quad \text{(Neumann)} \\ u_2 = u_1 \quad &\text{on} \quad \partial\Omega_2 \cap \partial\Omega_1 \quad \text{(Dirichlet)} \end{aligned}$$

OVERDETERMINED!


Open
808
(2 of 47)
200%
Ω_2
Ω_0
Ω_1
Fig. 1 Numerical approximation U of the solution of the classical Monopolist's problem (1), computed on a 50×50 grid. *Left* level sets of U, with $U = 0$ in *white*. *Center left* level sets of $\det(\nabla^2 U)$ (with again $U = 0$ in *white*); note the degenerate region Ω_1 where $\det(\nabla^2 U) = 0$. *Center right* distribution of products sold by the monopolist. *Right* profit margin of the monopolist for each type of product (margins are low on the one dimensional part of the product line, at the *bottom left*). *Color scales* on Fig. 10 (color figure online)
J.-M. Mirebeau (2016)
Springer



c.f. Boerma-Tsyvinski-Zimin 22+ blunt Ω_1^0 v targeted Ω_1^+ bunching

Free boundary problem

$u = u_i$ on Ω_i where

- on Ω_0 exclusion: $u_0 = 0$
- on Ω_1^0, Rochet-Choné's ODE: $u_1(x_1, x_2) = \frac{1}{2}k(x_1 + x_2)$ where

$$k(s) = \frac{3}{4}s^2 - as - \log|s - 2a| + const$$

subject to boundary conditions $k = 0$ and $k' = 0$ at lower boundary.

- on Ω_1^+, $u_1 = u_1^+$ given by a NEW system of ODE (for height $h(\cdot)$ and length $R(\cdot)$ of isochoice segments together with profile of $u_1^+(\cdot)$ along them), with boundary conditions

Free boundary problem

$u = u_i$ on Ω_i where

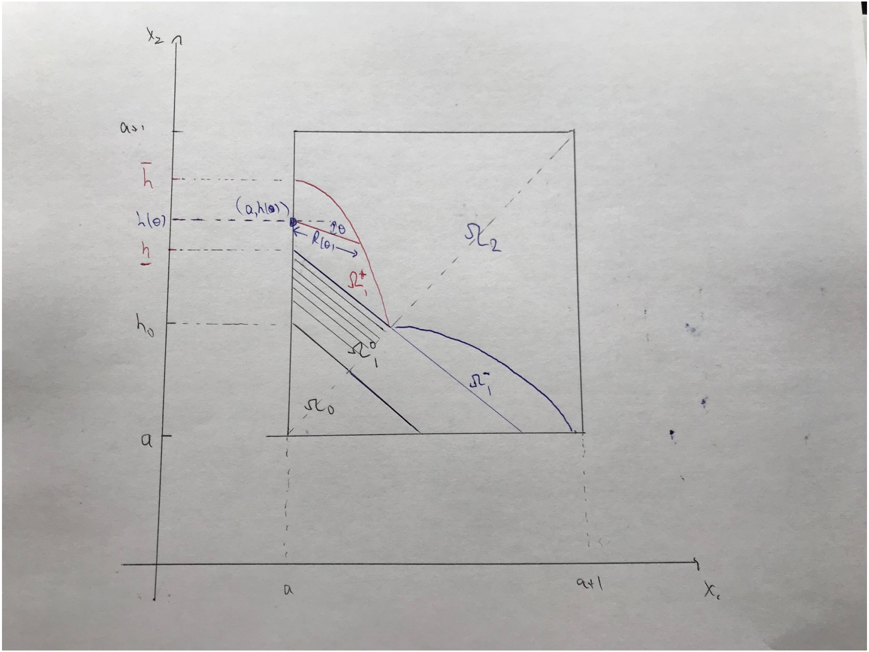
- on Ω_0 exclusion: $u_0 = 0$

- on Ω_1^0, Rochet-Choné's ODE: $u_1(x_1, x_2) = \frac{1}{2}k(x_1 + x_2)$ where
$$k(s) = \frac{3}{4}s^2 - as - \log|s - 2a| + const$$
subject to boundary conditions $k = 0$ and $k' = 0$ at lower boundary.

- on Ω_1^+, $u_1 = u_1^+$ given by a NEW system of ODE (for height $h(\cdot)$ and length $R(\cdot)$ of isochoice segments together with profile of $u_1^+(\cdot)$ along them), with boundary conditions $u_1^+(x_1, x_2) = k(x_1 + x_2)$ and $Du_1^+ = (k', k')$ on $\partial\Omega_1^0 \cap \partial\Omega_1^+$

- on Ω_2, PDE: $\Delta u_2 = 3$ with Rochet-Choné's overdetermined conditions

$$\begin{aligned}
(Du_2(x) - x) \cdot \hat{n}_{\Omega_2}(x) = 0 \quad &\text{on} \quad \partial X \cap \bar{\Omega}_2 \text{ and on } \{x_1 = x_2\} \\
(Du_2 - Du_1^+) \cdot \hat{n}_{\Omega_2}(x) = 0 \quad &\text{on} \quad \partial\Omega_2 \cap \partial\Omega_1^+ \quad \text{(Neumann)} \\
u_2 = u_1^+ \quad &\text{on} \quad \partial\Omega_2 \cap \partial\Omega_1^+ \quad \text{(Dirichlet)}
\end{aligned}$$



x_2
$a+1$
$\bar{h}$
$h(\theta)$
$\underline{h}$
h_0
a
$(a,h(\theta))$
2θ
$R(\theta)$
Ω_2
Ω_1^+
Ω_1^0
Ω_1^-
Ω_0
a
$a+1$
x_1

Precise Euler-Lagrange equation in the 'missing' region Ω_1^+

Index each isochoice segment in Ω_1^+ by its angle $\theta \geq -\frac{\pi}{4}$ to horizontal. Let $(a, h(\theta))$ denote its left-hand endpoint and parameterize the segment by distance $r \in [0, R(\theta)]$ to $(a, h(\theta))$. Along this segment of length $R(\theta)$,

$$u_1^+\Big((a, h(\theta)) + r(\cos\theta, \sin\theta)\Big) = m(\theta)r + b(\theta).$$

Precise Euler-Lagrange equation in the 'missing' region Ω_1^+

Index each isochoice segment in Ω_1^+ by its angle $\theta \geq -\frac{\pi}{4}$ to horizontal. Let $(a, h(\theta))$ denote its left-hand endpoint and parameterize the segment by distance $r \in [0, R(\theta)]$ to $(a, h(\theta))$. Along this segment of length $R(\theta)$,

$$u_1^+\Big((a, h(\theta)) + r(\cos\theta, \sin\theta)\Big) = m(\theta)r + b(\theta).$$

For $\underline{h} \in [a, a+1]$, $R : [-\frac{\pi}{4}, \frac{\pi}{2}] \to [0, a\sqrt{2})$ with $R(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(\underline{h} - a)$, solve

$$(m''(\theta) + m(\theta) - 2R(\theta))(m'(\theta)\sin\theta - m(\theta)\cos\theta + a) = \frac{3}{2}R^2(\theta)\cos\theta \quad (2)$$

$$m(-\tfrac{\pi}{4}) = 0, \qquad m'(-\tfrac{\pi}{4}) = \tfrac{1}{\sqrt{2}}k'(a + \underline{h}).$$

Precise Euler-Lagrange equation in the 'missing' region Ω_1^+

Index each isochoice segment in Ω_1^+ by its angle $\theta \geq -\frac{\pi}{4}$ to horizontal. Let $(a, h(\theta))$ denote its left-hand endpoint and parameterize the segment by distance $r \in [0, R(\theta)]$ to $(a, h(\theta))$. Along this segment of length $R(\theta)$,

$$u_1^+\Big((a, h(\theta)) + r(\cos\theta, \sin\theta)\Big) = m(\theta)r + b(\theta).$$

For $\underline{h} \in [a, a+1]$, $R : [-\frac{\pi}{4}, \frac{\pi}{2}] \to [0, a\sqrt{2})$ with $R(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(\underline{h} - a)$, solve

$$(m''(\theta) + m(\theta) - 2R(\theta))(m'(\theta)\sin\theta - m(\theta)\cos\theta + a) = \frac{3}{2}R^2(\theta)\cos\theta \quad (2)$$

$$m(-\tfrac{\pi}{4}) = 0, \qquad m'(-\tfrac{\pi}{4}) = \tfrac{1}{\sqrt{2}}k'(a + \underline{h}). \qquad \text{Then set} \quad (3)$$

$$h(t) = \underline{h} + \frac{1}{3}\int_{-\pi/4}^{t} (m''(\theta) + m(\theta) - 2R(\theta))\frac{d\theta}{\cos\theta}, \quad (4)$$

$$b(t) = \frac{1}{2}k(a + \underline{h}) + \int_{-\pi/4}^{t} (m'(\theta)\cos\theta + m(\theta)\sin\theta)h'(\theta)d\theta. \quad (5)$$

• for $\underline{h} \in [a, a+1]$, $R : [-\frac{\pi}{4}, \frac{\pi}{2}] \to [0, a\sqrt{2})$ Lipschitz (say) and $R(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(\underline{h} - a)$ we can solve (2)–(5) to find Ω_1^+ and u_+^1.

• we can then solve the resulting Neumann problem for $\Delta u_2 = 3$ on Ω_2

• while it is not yet *rigorously* proved is that some choice of $\underline{h}$ and $R(\cdot)$ also yields $u_1 - u_2 = const$ on $\partial\Omega_2 \setminus \partial X$,

- for $\underline{h} \in [a, a+1]$, $R : [-\frac{\pi}{4}, \frac{\pi}{2}] \to [0, a\sqrt{2})$ Lipschitz (say) and $R(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(\underline{h} - a)$ we can solve (2)–(5) to find Ω_1^+ and u_+^1.

- we can then solve the resulting Neumann problem for $\Delta u_2 = 3$ on Ω_2

- while it is not yet *rigorously* proved is that some choice of $\underline{h}$ and $R(\cdot)$ also yields $u_1 - u_2 = const$ on $\partial\Omega_2 \setminus \partial X$, we hope to do this in the future

- if a choice exists such that, absorbing the constant into u_2, the resulting u given by $u_i^{(\pm)}$ on $\Omega_i^{(\pm)}$ for $i \in \{0, 1, 2\}$ is in $\mathcal{U}$, our new duality can be used to certify that u is the desired optimizer

WHY DO WE EXPECT SUCH A CHOICE TO EXIST?

• for $\underline{h} \in [a, a+1]$, $R : [-\frac{\pi}{4}, \frac{\pi}{2}] \to [0, a\sqrt{2})$ Lipschitz (say) and $R(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(\underline{h} - a)$ we can solve (2)–(5) to find Ω_1^+ and u_+^1.

• we can then solve the resulting Neumann problem for $\Delta u_2 = 3$ on Ω_2

• while it is not yet *rigorously* proved is that some choice of $\underline{h}$ and $R(\cdot)$ also yields $u_1 - u_2 = const$ on $\partial\Omega_2 \setminus \partial X$, we hope to do this in the future

• if a choice exists such that, absorbing the constant into u_2, the resulting u given by $u_i^{(\pm)}$ on $\Omega_i^{(\pm)}$ for $i \in \{0, 1, 2\}$ is in $\mathcal{U}$, our new duality can be used to certify that u is the desired optimizer

WHY DO WE EXPECT SUCH A CHOICE TO EXIST?

• a unique optimizer $\bar{u} \in \mathcal{U}$ is known to exist (Rochet-Choné) and $\bar{u} \in C^{1,1}_{loc}(X^0)$ (Caffarelli-Lions); if the sets Ω_i where its Hessian is rank i are smooth enough, and Ω_1 has the expected 3 components, then (2)–(5) and the overdetermined Poisson problem $\Delta u_2 = 3$ must be satisfied

• but maybe Ω_i are not smooth enough, or Ω_1 is not (simply) connected and/or has more than three components $\Omega_1^0, \Omega_1^\pm$ (some too small for the numerics to resolve); we doubt this, but can't rule it out rigorously yet...

A regularity result: Lipschitz product selection

Theorem (M.-Rankin-Zhang 23+)

If b and $\tilde{b}(y,x) = b(x,y)$ both satisfy (B0-B3), c satisfies (C0-C2) and $d\mu(x) = f(x)dx$ with $\log f \in C^{0,1}$ then $u \in C^{1,1}_{loc}(X^0)$.

- map from buyer to product types is locally Lipschitz!
- extends Caffarelli-Lions '06+ to b & c non-quadratic
- improves Chen '13 from C^1_{loc} to $C^{1,1}_{loc}$
- sharp: examples for $n = 1 = m$ show $u \notin C^2_{loc}(X^0)$
- idea: use energetic comparison to pinch u between parabolas

Lemma (Geometry of a carefully chosen trial function)

Given $\delta > 0$, there exists $C_0, C_1, C_2 > 0$ such that if $u = u^{\tilde{b}b}$ is optimal and $d(x_0, \partial X) > \delta$ and $y_0 = \bar{y}_b(Du(x_0), x_0)$ then if $r < C_0$ and

$$h = \sup_{x \in B_r(x_0)} u(x) - [u(x_0) + b(x, y_0) - b(x_0, y_0)] > 0$$

then some $A(\cdot) = b(\cdot, y') + a'$ makes $S := \{x \in X \mid u < A\}$ a neighbourhood of x_0 (contained in a strip of width $2r$) with

$$\sup_{x \in S} A(x) - u(x) \leq h$$

and

$$0 \geq \frac{1}{|S|} \int_S \Big[c(y) - b(x, y) \Big]_{y=y'}^{y=\bar{y}(Du(x),x)} f(x) dx \geq -C_1 h + C_2 \frac{h^2}{r^2}.$$

CONCLUSIONS

- Concavity, when present, is a powerful tool for optimization
- for numerics, uniqueness, stability, and characterization of optimum
- Duality of price menu $v(y)$ with buyers' indirect utilities $u(x) = v^b(x)$
- Necessary and sufficient conditions for concavity of monopolist's problem (as a function of u)
- and a sharp smoothness result $u \in C^{1,1}_{loc}(X^0)$
- Related to curvature conditions governing regularity in optimal transport problems (à la Ma-Trudinger-Wang and Kim-McCann)
- new duality certifying solutions for bilinear benefit $b(x, y) = x \cdot y$
- square example requires solving an unexpected free boundary problem

CONCLUSIONS

- Concavity, when present, is a powerful tool for optimization
- for numerics, uniqueness, stability, and characterization of optimum
- Duality of price menu $v(y)$ with buyers' indirect utilities $u(x) = v^b(x)$
- Necessary and sufficient conditions for concavity of monopolist's problem (as a function of u)
- and a sharp smoothness result $u \in C^{1,1}_{loc}(X^0)$
- Related to curvature conditions governing regularity in optimal transport problems (à la Ma-Trudinger-Wang and Kim-McCann)
- new duality certifying solutions for bilinear benefit $b(x, y) = x \cdot y$
- square example requires solving an unexpected free boundary problem

THANK YOU!