## Causal differential calculus & d'Alembert comparison

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## A nonsmooth framework for gravity

- replace Lorentz manifold  $(M, g_{ij})$  of relativity with *metric spacetime* M (variant on Kunzinger-Sämann's '18 Lorentzian prelength spaces)
- $\ell: M^2 \longrightarrow \{-\infty\} \cup [0,\infty)$  is called a *time-separation* function if

$$\ell(x,y) + \ell(y,z) \le \ell(x,z) \qquad \forall x,y,z \in M$$

• *l* defines the *transitive* relations *causality* and *chronology*:

 $\leq := \{\ell \geq 0\} \qquad \ll := \{\ell > 0\}$ future  $J^+(x) = \{y \in M \mid y \geq x\} \qquad I^+(x) := \{y \in M \mid y \gg x\}$ past  $J^-(z) := \{y \in M \mid y \leq z\} \qquad I^-(z) := \{y \in M \mid y \ll z\}$ 

- assume  $\ell(y, y) = 0 \ \forall y \in M$ , so  $\leq$  also reflexive
- chronological topology: the coarsest topology with  $I^{\pm}(y)$  open  $\forall y \in M$

• a topology is called *Polish* if it has a complete, separable metrization

#### Definition (Metric spacetime; time-reversal)

A time-separation function  $\ell: M^2 \longrightarrow \{-\infty\} \cup [0,\infty)$  as above makes  $(M,\ell)$  a *metric spacetime* if the chronological topology it induces is Polish. The *time-reversal*  $(M,\ell^*)$  of  $(M,\ell)$  refers to  $\ell^*(y,x) = \ell(x,y)$ .

- metrizability implies  $\leq$  is partial-order: i.e.  $(x \leq z \& z \leq x) \Rightarrow (x = z)$
- $\leq$  is *forward-complete*  $\Leftrightarrow x_i \leq x_{i+1} \leq z (\forall i \in \mathbf{N})$  implies  $\lim_{i \to \infty} x_i$  exists

#### Definition (Forward spacetime — our standing assumption)

A metric spacetime  $(M, \ell)$  (with its causal and chronological relations  $\leq$  and  $\ll$  and Polish chronological topology) is called *forward* if the partial order  $\leq$  is forward-complete and  $\ell$  is upper semicontinuous.

- write  $(M, \ell)$  is *backward*  $\Leftrightarrow$  its time-reversal  $(M, \ell^*)$  is forward
- let  $J^+(X) := \cup_{x \in X} J^+(x)$  and  $J^-(Y) := \cup_{y \in Y} J^-(y)$

### Definition (Emeralds)

An *emerald* refers to  $J(X, Y) := J^+(X) \cap J^-(Y)$  with  $X, Y \subset M$  compact.

•  $(M, \ell)$  is called *globally hyperbolic* if every emerald is compact

### Example (Manifolds)

Globally hyperbolic Lorentzian length spaces are forward spacetimes (as are globally hyperbolic smooth Lorentzian manifold spacetimes).

### Example (Manifolds with boundary)

The interval  $\left[-1,1\right]$  with the time-separation

$$\ell(x,y) := egin{cases} y-x & ext{if } y \geq x, \ -\infty & ext{else}, \end{cases}$$

is a forward spacetime (but not a Lorentzian length space nor a manifold, whereas its open subset (-1, 1) is both a globally hyperbolic forward spacetime and a Lorentzian length space and manifold).

# Calculus of worldlines (i.e. nondecreasing curves)

### Definition (Causal curve and speed; c.f. [A90] for (M, d))

 $\sigma: [0,1] \longrightarrow M$  is *causal*  $\Leftrightarrow \sigma_s := \sigma(s) \leq \sigma(t)$  for all  $0 \leq s < t \leq 1$ ; (it is *timelike*  $\Leftrightarrow$  we can replace  $\leq$  above with  $\ll$ ). Its *causal speed* refers to the (pointwise) limit on (0,1)

$$|\dot{\sigma}(s)| := \lim_{h \downarrow 0} rac{\ell(\sigma_{s+h}, \sigma_s)}{h}$$

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in a metric (resp. forward) spacetime, discontinuities of a causal curve σ are countable (and σ may be taken left-continuous without loss, resp.)
the set LCC([0, 1]; M) of Left-Continuous Causal curves metrized by

$$D(\sigma,\tau) := d(\sigma_0,\tau_0) + \int_0^1 d(\sigma_s,\tau_s) ds$$

is Polish, where d makes the chronological topology Polish on  $(M, \ell)$ 

• Limit-curve theorem:  $C \subset M$  compact makes LCC([0,1]; C) D-compact

## q-Lagrangian action and (rough) $\ell$ -geodesics

## Definition (q-Lagrangian action; geodesics; c.f. [M.20] [MS23])

Given  $0 \neq q < 1$ , the *action* of a causal curve refers to

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Causal curves *maximizing* this action (for given endpoints) are called *rough geodesics*; if  $\sigma \in LCC([0, 1]; M)$  then simply *geodesic*.

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- recall twin paradox
- maximizers are independent of q;
- the set of geodesics is denoted CGeo(M);
- curves in  $TGeo(M) := \{ \sigma \in CGeo(M) \mid A_q[\sigma] > 0 \}$  are called *timelike* or  $\ell$ -geodesics.

# Midpoint sets and bi-regularity; characterizing geodesics

Lemma (Indpendence of q; affine parameterization)

A curve  $\sigma : [0,1] \longrightarrow M$  is a rough  $(\ell$ -)geodesic iff for all  $0 \le s < t \le 1$ ,

 $\ell(\sigma(s),\sigma(t)) = (t-s)\ell(\sigma(0),\sigma(1)) \quad (>0).$ 

For all  $x, y \in M$  and  $s \in [0, 1]$  define the midpoint set

 $Z_s(x,y) = \{z \in M \mid \ell(x,z) = s\ell(x,y), \quad \ell(z,y) = (1-s)\ell(x,y)\}.$ 

Definition (right-, left-, and bi-regularity c.f. [M.24])

A metric spacetime  $(M, \ell)$  is called *right-regular* if  $Z_0(x, y) = \{x\}$ ,

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Definition (right-, left-, and bi-regularity c.f. [M.24])

A metric spacetime  $(M, \ell)$  is called *right-regular* if  $Z_0(x, y) = \{x\}$ , *left-regular* if  $Z_1(x, y) = \{y\}$ , and *bi-regular* if both hold  $(\forall x \ll y \in M)$ .

- in left-regular forward spacetime, any rough  $\ell$ -geodesic is left-continuous.
- however bi-regularity need not imply countinuity of rough geodesics, unless the spacetime is backward as well as forward.

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## Fuzzy events: lifting the geometry from events to measures

Optimal transport:

$$\ell_q(\mu,\nu) := \sup_{\gamma \in \mathsf{F}_{\leq}(\mu,\nu)} \left( \int_{M^2} \ell(x,y)^q d\gamma(x,y) \right)^{1/q}$$

defines a time-separation (and a causal relation [EM17]) between Borel probability measures  $\mu, \nu \in \mathcal{P}_{em}(M)$  on emeralds in M. Here

$$\begin{split} \Gamma_{\leq}(\mu,\nu) &:= \left\{ \gamma \geq 0 \text{ on } M^2 \mid \gamma[\{\ell \geq 0\}] = 1, \quad \mu[Y] = \gamma[Y \times M] \\ \forall Y \subset M, \qquad \gamma[M \times Y] = \nu[Y] \right\} \end{split}$$

- maximizers  $\gamma$  exist if  $\Gamma_{\leq}(\mu, \nu) \neq \emptyset$  and are called *q*-optimal couplings
- the  $\ell_q$ -speed along any causal curve  $(\mu_s)_{s\in[0,1]}$  of measures is

$$|\dot{\mu}_{s}|_{q} := \lim_{h \downarrow 0} \frac{\ell_{q}(\mu_{s}, \mu_{s+h})}{h}$$

# Tangent fields; lifting curves $(\mu_t)_t$ to measures $\pi$ on curves

#### Definition (Rough $\ell_q$ -geodesics can be defined like rough $\ell$ -geodesics)

Given 0 
eq q < 1, the action of a causal curve  $(\mu_t)_{t \in [0,1]} \subset \mathcal{P}(M)$  is

$$\mathcal{A}_{q}[\mu] := \frac{1}{q} \int_{0}^{1} |\dot{\mu}_{t}|_{q}^{q} dt \leq \frac{1}{q} \ell_{q}(\mu_{0}, \mu_{1})^{q} < \infty \text{ if } \mu_{0}, \mu_{1} \in \mathcal{P}_{em}(M).$$

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Theorem (Lifting curves of measures in forward spacetimes c.f.[Lis07])

Conversely, if  $(\mu_t)_{t \in [0,1]} \subset \mathcal{P}(M)$  is causal, narrowly left-continuous on [0,1], and tight on  $(\epsilon, 1-\epsilon)$  ( $\forall \epsilon > 0$ ) then it's induced by a plan  $\pi \in \mathcal{P}(LCC([0,1];M))$  with expected action

$$\int A_q[\sigma] d\pi(\sigma) = \mathcal{A}_q[\mu]$$

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$$\int A_q[\sigma] d\pi(\sigma) = \mathcal{A}_q[\mu] \quad (= \ell_q(\mu_0, \mu_1)^q / q \text{ if } \pi \text{ is "q-optimal"})$$

 $\bullet$  these measures  $\pi$  on curves (i.e. 'plans') represent tangent fields

Corollary (Optimal plans concentrate on geodesics)

If  $\pi \in \mathcal{P}(LCC([0, 1]; M))$  is q-optimal, then  $\pi[CGeo] = 1$ .

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Corollary (Narrow forward-completeness in a forward spacetime)

If  $\mu_i \leq \mu_{i+1} \leq \nu$  in  $(\mathcal{P}(M), \ell_q)$ , then  $\lim_{i \to \infty} \mu_i$  converges narrowly.

• plays a crucial role in our eventual construction of 'good' test plans

## Definition (Strict timelike q-dualizability; c.f. [M.20] [CM24])

The pair  $\mu, \nu \in \mathcal{P}_{em}(M)$  is *strictly timelike q-dualizable* iff every *q*-optimal coupling  $\gamma \in \Gamma_{\leq}(\mu, \nu)$  vanishes outside  $\{\ell > 0\}$ .

Lemma (Narrow continuity of rough  $\ell_{q}$ -geodesics)

If  $(M, \ell)$  is a bi-regular, forward spacetime

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### Lemma (Narrow continuity of rough $\ell_{q}$ -geodesics)

If  $(M, \ell)$  is a bi-regular, forward spacetime and  $(\mu_t)_{t \in [0,1]}$  is a rough  $\ell_q$ -geodesic with strictly timelike q-dualizable endpoints  $\mu_0, \mu_1 \in \mathcal{P}_{em}(M)$ , then  $t \in [0, 1] \mapsto \mu_t$  is narrowly continuous wherever it is locally tight.

• local tightness can come from e.g., global hyperbolicity or density bounds or narrow forward-completeness...

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# (Exact, future-directed) cotangent fields; their magnitudes

Definition (Causal functions (nondecreasing); form a convex cone)

 $f: M \longrightarrow [-\infty, \infty]$  is causal  $\Leftrightarrow \ell(x, y) \ge 0$  implies  $f(x) \le f(y)$ .

### Definition (Metric-measure spacetimes; test plan; maximal subslope)

Fix a Borel measure *m* on  $(M, \ell)$  assigning finite mass to each emerald. A plan  $\pi \in \mathcal{P}(LCC([0, 1]; M))$  is called (initially) *test*  $\Leftrightarrow$ 

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$$f(\sigma_1) - f(\sigma_0) \geq \int_0^1 g(\sigma_t) |\dot{\sigma}_t| dt$$

for every test plan  $\pi$  and  $\pi$ -a.e. curve  $\sigma$ . They form a stable lattice. Each *m*-measurable causal *f* admits a *maximal weak subslope*, denoted g = |df|.

• this very general definition, c.f. [AGS14], good for integration-by-parts

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# Infinitesimal Minkowskianity

#### Lemma (Examples of weak subslopes; TMCP $\Rightarrow$ Lebesgue theorem)

Continuity of causal f and  $\ell_+ = max\{\ell,0\}$  imply m-a.e. y satisfies

$$\liminf_{x\ll y} \frac{f(y)-f(x)}{\ell(x,y)} \leq |df(y)|, \qquad \liminf_{z\gg y} \frac{f(z)-f(y)}{\ell(y,z)} \leq |df(y)|.$$

## Definition (c.f. infinitesimally Hilbertian [G15] rather than [AGS14d])

A metric-measure spacetime  $(M, \ell, m)$  is *infinitesimally Minkowskian*  $\Leftrightarrow$ 

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A metric-measure spacetime  $(M, \ell, m)$  is *infinitesimally Minkowskian*  $\Leftrightarrow$  all real causal *m*-measurable functions f, g satisfy the parallelogram law

$$|d(f+g)|^2 + |dg|^2 = 2|d(f+2g)|^2 + 2|df|^2$$
 m-a.e.

• equivalently, the following polarization is positively bilinear *m*-a.e.:

$$2((df, dg)) := |d(f+g)|^2 - |df|^2 - |dg|^2$$

• distinguishes Lorentz from Lorentz-Finsler metrics on e.g.  $\mathbf{R}^{n}$  [BO24]

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## Convex analysis; horizontal derivatives; raising indices

Just as causal curves and functions on a smooth Lorentz manifold satisfy

$$\langle df, \dot{\sigma} \rangle \ge \frac{1}{p} \| df \|_*^p + \frac{1}{q} \| \dot{\sigma} \|^q \quad \text{when } p^{-1} + q^{-1} = 1$$

with equality iff  $\langle \dot{\sigma}, \cdot \rangle = \|df\|_*^{p-2} df(\cdot)$ , i.e. iff  $\dot{\sigma} = \|\nabla f\|^{p-2} \nabla f$  [M.20],

Theorem (Nonsmooth Fenchel-Young inequality for  $0 \neq q < 1$ )

If  $(e_s)_{\#}\pi \to (e_0)_{\#}\pi$  narrowly,  $|df|^p \in L^1((e_0)_{\#}\pi)$ , and  $\pi$  initially test then

$$\lim_{s\downarrow 0} \int \frac{f(\sigma_s) - f(\sigma_0)}{s} d\pi(\sigma) \ge$$

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• limit on left called *horizontal* (inner, Lagrangian) derivative of f along  $\pi$ 

• aims at bilinear pairing of  $\pi$  with f; (NB concave p-Dirichlet energy of f)

#### Definition (Identifying tangent with cotangent fields)

If  $\lim_{s\downarrow 0}$  exists and equality holds, we say  $\pi$  represents the *p*-gradient of *f*. A nonlinear duality between some tangent and cotangent fields ( $\pi$  and *f*)

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# Perturbation & variational derivative of *p*-Dirichlet energy

• given *m*-measurable  $E \subset M$ , write  $g \in Pert_p(f, E)$  if for all  $\epsilon > 0$  small enough,  $f + \epsilon g$  is causal and  $|d(f + \epsilon g)|^p \in L^1(E, dm)$ .

Theorem (Horizontal dominates vertical derivative; c.f. [G15])

If  $f : M \longrightarrow \overline{\mathbf{R}}$  is causal,  $g \in Pert_p(f, E)$ , and  $\pi$  represents the p-gradient of f and is concentrated on curves remaining initially in E, then

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variation of *p*-Dirichlet energy =:  $\int d^+g(\nabla f)|df|^{p-2}d(e_0)_{\#} \pi$ 

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writing of *p*-Dirichlet energy =: 
$$\int d^+g(\nabla f) |df|^{p-2} d(e_0)_{\#} \pi$$

last is direction g vertical (/ outer / Eulerian) derivative of p-energy at f
nonlinear in f but becomes linear in g if two-sided limit in ε exists

Corollary (If  $(M, \ell, m)$  is infinitesimally Minkowskian)and if  $-g, g \in Pert_p(f, E)$  then  $\lim_{\epsilon \to 0}$  and  $\lim_{s \downarrow 0}$  exist & equality holds above!Robert J McGam (Toronto)Nonsmooth gravity/d'Alembert comparison24 September 202415/24

## Curvature bounds via entropy

Given  $N \in (1, \infty)$ , define the N-Renyi (power-law) entropy of  $\mu \in \mathcal{P}(M)$  by

$$S_N(\mu) := -\int_M (\frac{d\mu}{dm})^{1-1/N} dm$$

• in the smooth globally hyperbolic setting, convexity properties of  $t \in [0, 1] \mapsto S_N(\mu_t)$  along  $\ell_q$ -geodesics are well-known to characterize timelike lower Ricci curvature bounds [B23] [MS23] [M.20]; c.f. [RS04][CMS01][OV00][M.94]

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*TMCP*<sup> $\pm$ </sup>: a poor man's version of lower Ricci curvature bounds

- we impose only sublinearity of  $S_N(\mu_t)$  only along  $\ell_q$ -geodesics starting or ending at a Dirac point mass — the *timelike measure contraction* properties  $TMCP^{\pm}$  of [B23]; c.f. [CM24] [LV09] [O07] [S06]
- if  $(\mu_0, \delta_z)$  are strictly timelike *q*-dualizable precisely one  $\ell_q$ -geodesic links  $\mu_0$  to  $\delta_z$ ; moreoever  $S_N(\delta_z) = 0$

#### Definition (Future timelike measure contraction property; c.f. [B23])

For  $K \in \mathbb{R}$  write  $(M, \ell, m) \in TMCP^+(K, N)$  if  $\forall \mu_0 \in \mathcal{P}_{em}(M) \cap L^1(m)$ and each  $z \in \text{spt } m$  with  $\mu_0[I^-(z)] = 1$ , for some (hence all)  $0 \neq q < 1$ , there exists a (rough)  $\ell_q$ -geodesic from  $\mu_0$  to  $\mu_1 := \delta_z$  such that all  $t \in [0, 1]$  and  $N' \geq N$  satisfy

$$S_{N'}(\mu_t) \leq -\int \tau_{K,N}^{(1-t)}(\ell(x,z)) \frac{d\mu_0}{dm}(x)^{1-1/N'} dm(x).$$

Past version:  $(M, \ell, m) \in TMCP^{-}(K, N) \Leftrightarrow (M, \ell^*, m) \in TMCP^{+}(K, N)$ .

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$$\tau_{0,N}^{(1-t)}(\ell) := 1 - t$$
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•  $\tau_{0,N}^{(1-t)}(\ell) := 1 - t$  for K = 0; asserts sublinearity of  $t \in [0, 1] \mapsto S_{N'}(\mu_t)$ , and follows from the strong energy condition, a case of primary interest

• a smooth globally hyperbolic Lorentzian manifold  $M^n$  satisfies  $TMCP^{\pm}(K, N)$  if  $n \leq N$  and  $Ric(v, v) \geq Kg(v, v)$  for all timelike  $v \in TM$ 

## Test plans: finding $\ell_q$ -geodesics having density bounds

#### Theorem (Initial test plans with Dirac targets; c.f. [B23][CM17][R13])

Fix  $(K \in \mathbb{R} \text{ or}) K = 0 \neq q < 1 < N < \infty$ , a bi-regular forward spacetime  $(M, \ell, m) \in TMCP^+(K, N)$  and  $z \in M$ . If  $\mu_0[I^-(z)] = 1$  for  $\mu_0 \in L^{\infty}(m) \cap \mathcal{P}_{em}(M)$  then there exists a q-optimal plan  $\pi$  inducing (an  $\ell_q$ -geodesic)  $\mu_t := (e_t)_{\#}\pi$  from  $\mu_0$  to  $\mu_1 := \delta_z$  such that  $t \in [0, 1] \mapsto S_{N'}(\mu_t)$  is (suitably) sublinear for each  $N' \geq N$  and

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$$\|\frac{d\mu_t}{dm}\|_{L^{\infty}(m)} \leq \frac{c_{\mathcal{K},\mathcal{N},\ell}}{(1-t)^{\mathcal{N}}} \|\frac{d\mu_0}{dm}\|_{L^{\infty}(m)}.$$

- $c_{0,N,\ell} = 1$  if K = 0 (else  $c_{K,N,\ell} := \exp(t \|\ell\|_{L^{\infty}(\mu_0 \times \mu_1)} \sqrt{K_{-}(N-1)})$ )
- lack of compactness overcome with order-theoretic ideas of [G+][G04]
- extends to non-Dirac targets provided

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- lack of compactness overcome with order-theoretic ideas of [G+][G04]
- extends to non-Dirac targets provided  $(\mu_0, \mu_1)$  strictly timelike q-dualizable and  $(M, \ell, m)$  is (q-essentially) timelike nonbranching,

COROLLARY (Busemann and Lorentz distance functions have unit slope)  $g(\cdot) = -\ell(\cdot, z)$  satisfies |dg| = 1 *m*-a.e. on  $I^-(z)$ 

## When is the *p*-gradient of *f* represented by a test plan $\pi$ ?

$$f^{(q)}(z) := \sup_{x \in I^{-}(z)} f(x) + \frac{\ell(x, z)^{q}}{q} \qquad g_{q}(x) := \inf_{z \in I^{+}(x)}$$

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- write  $f: M \longrightarrow \overline{\mathbf{R}}$  is  $\frac{\ell^q}{q}$ -concave if  $f = g_q$  for some  $g: M \longrightarrow \overline{\mathbf{R}}$ ;
- then f is causal, upper semicontinuous, and  $\partial_{\ell^q/q} f$  relatively closed in  $\ll$

$$\partial_{\ell^q/q} f := \{ x \ll z \mid f^{(q)}(z) = f(x) + \frac{\ell(x,z)^q}{q} \in \mathbf{R} \} \subset M^2, \text{ if } \ell_+ \in \mathcal{C}(M)$$

#### Theorem (A metric Brenier-M. thm; cf.[CM24][MS23][M.20][AGS14])

Fix  $0 \neq q < 1$  and  $p^{-1} + q^{-1} = 1$ . Let  $(M, \ell, m)$  be forward,  $\ell_+$  continuous and  $f = (f^{(q)})_q$ .

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$$|df|(\sigma_0) = \ell(\sigma_0, \sigma_1)^{q-1}.]$$

$$\operatorname{dom} \partial_{\ell^q/q} f := \{ x \in M \mid \partial_{\ell^q/q} f \cap (\{x\} \times M) \neq \emptyset \}$$

#### Theorem (d'Alembert comparison theorem: $\Box_p f \leq N$ if K = 0)

Fix  $0 \neq q < 1 = p^{-1} + q^{-1} < N < \infty$ , a bi-regular forward spacetime  $(M, \ell, m) \in TMCP^+(K, N)$  with  $\ell_+ \in C(M)$ ,  $K \in \mathbb{R}$  and  $f = (f^{(q)})_q$ . Let  $(M, \ell, m)$  be (q-essential) timelike nonbranching unless  $\exists z \in M$  with

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If  $0 \le \phi \in Pert_p(f) \cap L^{\infty}$ , compact support and  $m[\operatorname{spt} \phi \setminus \operatorname{dom} \partial_{\ell^q/q} f] = 0$ then

$$\int_{M} d^{+}\phi(\nabla f) |df|^{p-2} dm \leq \int_{M} \tilde{\tau}_{K,N}(|df|^{p-1})\phi dm$$

$$\tilde{\tau}_{K,N}(\mathbf{r}) := N \frac{\partial \tau_{K,N}^t(\mathbf{r})}{\partial t|_{t=1}} = \begin{cases} N & \text{if } \mathbf{K} = \mathbf{0} \\ 1 + r\sqrt{(N-1)|K|} \cot(r\sqrt{\frac{K}{N-1}}) & \text{else.} \end{cases}$$

#### Corollary

In the same nonsmooth sense and setting, we establish a chain rule yielding

$$\Box_{
ho}(-\ell(\cdot,z)) \leq rac{N-1}{\ell(\cdot,z)} \quad ext{ on } I^-(z) \; .$$

• Analogous results also hold true in backward spacetimes. After time-reversing them, the forward  $(M, \ell, m) \in TMCP^{-}(K, N)$  satisfies

$$\Box_p(\ell(\mathbf{x},\cdot)) \geq -\frac{N-1}{\ell(\mathbf{x},\cdot)} \quad \text{on } I^+(\mathbf{x})$$

- It is conceivable that  $Pert_{\rho}(f, M)$  is sometimes too sparse to be of use
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#### Corollary

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- It is conceivable that  $Pert_{\rho}(f, M)$  is sometimes too sparse to be of use
- However, in smooth globally hyperbolic spacetimes,  $Pert_p(f, M)$  is rich enough to imply the preceding conclusions in the usual, distributional sense
- Eschenburg (1988) proved such estimates hold where  $\ell(\cdot,z)$  is smooth
- our results extend his across the timelike cutlocus for the first time

• Calabi (1958) proved 2-Laplacian comparison and extended it across the Riemannian cutlocus; (his formulation and technique foreshadowed the theory of viscosity solutions, but are utterly different from ours)

- thus even on smooth globally hyperbolic manifolds we obtain new results
- functional analysis:  $\Box_q f$  is a measure, but nonunique unless infinitesimal Minkowskianity  $LTMCP^{\pm}(K, N)$  holds and  $Pert_p(f, E)$  is dense; c.f. [G15]

• localization: [B24+] establishes many fundamental properties of  $\Box_p$  by developing an approach based on needle decompositions; c.f. [CM20]

# Selected references (apologies for omissions/oversights):

Beran-Braun-Calisto-Gigli-M.-Ohanyan-Rott-Sämann arXiv:2408.15968 Octet

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#### Beran-Braun-Calisto-Gigli-M.-Ohanyan-Rott-Sämann arXiv:2408.15968 Octet

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#### THANK YOU!