A nonsmooth approach to Einstein's theory of gravity

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but often predicts such manifolds are geodesically incomplete.

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a nonmsooth theory is highly desirable

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Example (Inspiring positive signature developments)

In metric(-measure) geometry with positive signature, there are theories of

• sectional curvature bounds based on triangle comparison (Aleksandrov. . .)

• pointed Gromov-Hausdorff limits of manifolds under lower Ricci and upper dimensional bounds (Fukaya, Gromov, Cheeger-Colding, . . .)

• Ricci lower bounds via displacement convexity of entropy (Bakry-Emery, Lott-Sturm-Villani, Ambrosio-Gigli-Savare, . . .)

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Can something similar be done in Lorentzian geometry?

• tidal forces (Kunzinger-Sämann '18)

• convergence of spaces (Müeller 22+, Minguzzi-Suhr 22+)

• Einstein eq (M 20, Mondino-Suhr 23, Cavalletti-Mondino 20+, Braun

Elliptic v hyperbolic geometry (c.f. BBCGMORS octet)

ELLIPTIC: $\mathbf{R}^n$ equipped with Euclidean norm $\|v\|_E := (\sum v_i^2)^{1/2}$

- $\|v+w\|_E \leq \|v\|_E + \|w\|_E$

HYPERBOLIC: Minkowski space $\mathbf{R}^n$ equipped with the *hyperbolic 'norm'*

$$\|v\|_F := \begin{cases} (v_1^2 - \sum_{i\geq 2} v_i^2)^{1/2} & v \in F := \left\{ v \in \mathbf{R}^n \mid v_1 \geq (\sum_{i \geq 2} v_i^2)^{1/2} \right\} \\ -\infty & \textit{else} \end{cases}$$

- $\|v+w\|_F \geq \|v\|_F + \|w\|_F$

the future $F \subset \mathbf{R}^n$ is a convex cone; $v \in F$ called *causal* or *future-directed*

- v is *timelike* if $v \in F \setminus \partial F$
- v is *lightlike (or null)* if $v \in \partial F \setminus \{0\}$

(• v is *spacelike* iff $\pm v \notin F$ and *past-directed* if $-v \in F$)

- smooth curves are called *timelike (etc.)* if all tangents are timelike (etc.)

A crash course in differential geometry: action principles

Manifold M^n with symmetric nondegenerate smooth tensor field $g_{ij} = g_{ji}$

RIEMANNIAN: $(g_{ij}) > 0$ defines Euclidean norm on each tangent space
- its geometry is also encoded in the (symmetric) distance function

$$d(x,y)^q := \inf_{\sigma(0)=x,\sigma(1)=y} \int \|\dot\sigma_t\|^q_{E_g} dt \qquad q > 1$$

LORENTZIAN: $g \sim (+1, -1, \ldots, -1)$ defines hyperbolic norm on $T_x M$
- its asymmetric geometry is also encoded in the time-separation function

$$\ell(x,y)^q := \sup_{\sigma(0)=x,\sigma(1)=y} \int \|\dot\sigma_t\|^q_{F_g} dt \qquad 0 < q < 1$$

- $(-\infty)^q := -\infty$ so $\ell(x,y) = -\infty$ unless a causal curve links x to y
- extremizers are independent of q; they are called *geodesics*
- $\ell(x,z) \geq \ell(x,y) + \ell(y,z)$ (analog of the triangle inequality d satisfies)

Concave p-energy: trading linearity for ellipticity

Additional conditions are imposed to ensure $\ell \neq +\infty$ and extremizers exist
• complete or *proper* (boundedly compact) in the Riemannian case
• *globally hyperbolic* in the Lorentzian case (i.e. compact diamonds, future F varies continuously over M, no closed future-directed curves)

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Concave Hamiltonian $H(w) = \frac{1}{p}\|w\|_{F^*}^p$ and Lagrangian $L(v) = \frac{1}{q}\|v\|_F^q$ satisfy $DH = (DL)^{-1}$ if $p^{-1} + q^{-1} = 1$ (here $p < 0$ since $0 < q < 1$)
- note $-L = (-H)^*$ jumps to $+\infty$ across future cone boundary ∂F (but $-H$ diverges continuously at the boundary of the dual cone F^*)
- nonsmooth already on smooth Lorentzian manifolds

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Beran Braun Calisti Gigli M. Ohanyan Rott Sämann (octet):
extremizers of p-Dirichlet energy $u \mapsto \int_M H(du)d\mathrm{vol}_g$ rel. to compactly supported perturbations satisfy a new degenerate elliptic nonlinear PDE
• trade linearity of d'Alembertian for ellipticity of p-d'Alembertian!
• gives new (elliptic) approach to Eschenburg-Galloway splitting theorem

The Riemann curvature tensor

Given (timelike) geodesics $(\sigma_s)_{s\in[0,1]}$ and $(\tau_t)_{t\in[0,1]}$ with $\sigma_0 = \tau_0$ and $\dot\sigma_0 - \dot\tau_0 \in F \setminus \partial F$,

$$\ell(\sigma_s, \tau_t)^2 = \|s\dot\sigma_0 - t\dot\tau_0\|^2_{F_g} - \frac{Sec}{6} s^2 t^2 + O((|s| + |t|)^5)$$

where sectional curvature $Sec = R(\dot\sigma_0, \dot\tau_0, \dot\sigma_0, \dot\tau_0)$ is quadratic in $\dot\sigma_0 \wedge \dot\tau_0$ and measures the leading order correction to Pythagoras

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- polarization of this quadratic form gives the *Riemann* tensor $R(\cdot,\cdot,\cdot,\cdot)$
- its trace $\mathrm{Ric}_{ik} = g^{jl} R_{ijkl}$ yields the *Ricci* tensor; $\mathrm{Ric}(v, v)$ measures the correction to Pythagoras averaged over all triangles including side v
- second trace $R = g^{ik}\mathrm{Ric}_{ik}$ yields the *scalar curvature*; in the elliptic case it gives leading order correction to the area of a sphere of radius r (and the volume of a ball of radius r)
- $d\mathrm{vol}_g(x) = \sqrt{|\det(g)|}\, d^n x$ in coordinates; (in the Riemannian case it coincides with the n-dimensional Hausdorff measure associated to d)

General relativity: Einstein's gravity and field equation

Gravity not a force, but rather a manifestation of curvature of spacetime
"Spacetime tells matter how to move" (along timelike/null geodesics...)

Field equation "Matter tells spacetime how to bend"

$$\begin{aligned} \textit{geometry} &= \textit{physics} \\ \text{curvature} &= \text{flux of energy and momentum} \end{aligned}$$

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$$\begin{aligned} \textit{geometry} &= \textit{physics} \\ \text{curvature} &= \text{flux of energy and momentum} \\ \mathrm{Ric}_{ij} - \frac{1}{2} R g_{ij} &= 8\pi T_{ij} \qquad \text{(replaces } \Delta\phi = \rho \text{ and } F = -\nabla\phi) \end{aligned}$$

• just integrate this local conservation law for $T_{ij}(x)$ to find $g_{ij}\ldots$

What if matter distribution is unknown?

Energy conditions and singularity theorems

WEC (weak energy condition): $T(v, v) \geq 0$ for all future $v \in F$ (physical)
SEC (strong energy condition): $\mathrm{Ric}(v, v) \geq 0$ for all future $v \in F$ (less ")
NEC (null energy condition): " ≥ 0 for all lightlike $v \in \partial F$

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[Cosmological constant (dark matter): $\geq (n-1)Kg(v,v)$]

Hawking '66 (big bang type) singularity theorem:
SEC + mean curvature bound $H_\Sigma \geq h > 0$ on a suitable hypersurface Σ implies finite-time singularities along all timelike geodesics through Σ
Cavalletti-Mondino '20+: genuinely nonsmooth version

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Kasue '83: $\Omega \subset \mathbf{R}^n$ with $H_{\partial\Omega} > h > 0$ bounds radius of largest ball in Ω
Burtscher-Ketterer-M.-Woolgar '20: extend to $CD(K, N)$ setting; in $RCD(K, N)$ setting equality only if $\Omega =$ ball or cone

Penrose '65 (stellar collapse type) singularity theorem
NEC + trapped codimension-2 compact surface S + suitable noncompact hypersurface Σ imply finite-time singularity along some null geodesic
Open: genuinely nonsmooth version?

A nonsmooth null energy condition

(Idea: reformulate the null energy condition in a timelike way

Lemma

Any smooth Riemannian manifold admits $k \in C(M)$ such that $\mathrm{Ric}(v, v) \geq k(x)g(v, v)$ for all $v \in T_x M$.

Proof.

$k(x) = \inf_{v \in T_x M} \frac{\mathrm{Ric}(v,v)}{g(v,v)}$. □

RMK: Can't hold in Lorentzian setting, even for $v \in F$, unless NEC holds.

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Theorem ((M. 23+) Not only sufficient, but necessary)

NEC $\Leftrightarrow \exists k \in C(M)$ such that every timelike vector $v \in T_xM$ satisfies

$$\mathrm{Ric}(v,v) \geq k(x)g(v,v).$$

i.e. NEC holds iff manifold admits a variable lower bound on timelike Ricci

What about the nonsmooth setting?

(please forget the foregoing)

Definition (Time-separation function)

On a set M of events, a *time-separation function* refers to $\ell : M \times M \longrightarrow \{-\infty\} \cup [0, \infty)$ satisfying the reverse triangle inequality and antisymmetry: $\forall x, y, z \in M$

$$\ell(x,y) \geq \ell(x,z) + \ell(z,y) \tag{1}$$

$$\min\{\ell(x,y), \ell(y,x)\} > -\infty \Leftrightarrow x = y. \tag{2}$$

Remark: (1) + (2) $\Rightarrow \ell(x,x) = 0$; (2) gives the arrow of time

Example (Minkowski space)

$M = R^{1,3}$ with $\ell(x,y) = \|y - x\|_F$

Example (Smooth globally hyperbolic Lorentzian manifolds)

Example (Causal spaces $(M, \leq, \ll)$ à la Kronheimer and Penrose '67)

A time-separation function gives a nested partial order $\leq$ and preorder $\ll$

$$M^2_{\leq} = \{(x,y) \in M^2 \mid \ell(x,y) \geq 0\}$$
$$M^2_{\ll} = \{ \qquad '' \qquad \mid \ell(x,y) > 0\}$$

Definition (Causal & timelike futures; causal diamonds and emeralds)

We say y lies in the *causal future* of x and write $x \leq y$ if $\ell(x,y) \geq 0$; we say y lies in the *timelike future* of x and write $x \ll y$ if $\ell(x,y) > 0$. Also

$$J^+(x) = \{y \in M \mid \ell(x,y) \geq 0\} = \textit{future} \qquad J^+(X) = \cup_{x \in X} J^+(x)$$
$$J^-(z) = \{y \in M \mid \ell(y,z) \geq 0\} = \textit{past} \qquad J^-(Z) = \cup_{z \in Z} J^-(z)$$
$$J(x,z) = J^+(x) \cap J^-(z) = \textit{diamond} \qquad J(X,Z) = J^+(X) \cap J^-(Z) = \textit{emerald}$$

and similarly $I^{\pm}(y)$ and $I(X,Z)$ but with strict inequalities $\ell > 0$.

Definition (Causal and timelike paths)

A path $s \mapsto \sigma(s) \in M$ is called *causal* if and only if $\ell(\sigma(s), \sigma(t)) \geq 0$ for all $s \leq t$, and *timelike* if and only if $\ell(\sigma(s), \sigma(t)) > 0$ for all $s < t$.

Definition (Lorentzian length of a causal path)

The (*negative*) ℓ-length of a causal path $\sigma : [a, b] \longrightarrow M$ is defined by

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The (*negative*) ℓ-length of a causal path $\sigma : [a, b] \longrightarrow M$ is defined by

$$\begin{aligned} L_{-\ell}(\sigma) &:= \sup_{k \in \mathbf{N}} \sup_{a=t_0 \leq t_1 \leq \cdots \leq t_k = b} - \sum_{i=1}^{k} \ell(\sigma(t_{i-1}), \sigma(t_i)) \\ &\geq -\ell(\sigma(a), \sigma(b)) \end{aligned}$$

by the triangle inequality.

Definition (ℓ-path)

A path $\sigma : [0,1] \longrightarrow M$ is called an ℓ-path if and only if

$$\ell(\sigma(s),\sigma(t)) = (t-s)\ell(\sigma(0),\sigma(1)) > 0 \qquad \forall 0 \le s < t \le 1.$$

We denote the set of ℓ-paths by $\mathrm{TPath}^{\ell}(M)$.

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Definition

(M, ℓ) is *timelike ℓ-path space* if each $x \ll y$ are endpoints of an ℓ-path.

- Kunzinger and Sämann's *regular globally hyperbolic Lorentzian length spaces* provide a rich class of examples of timelike ℓ-path spaces
- to achieve this, they need a (metrizable) topology

a variation on Kunzinger & Sämann (hereafter K-S)

Definition (Metric spacetime)

A metric space (M, d) equipped with its metric topology and a time-separation function ℓ is called a *metric spacetime*

Definition (Causal curve)

A nonconstant causal *path* is called a causal *curve* if it is d-Lipschitz.

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Definition (Non-total imprisoning)

A metric spacetime (M, d, ℓ) is *non-total imprisoning* if each compact $K \subset M$ has a bound $\sup L_d(\sigma) < \infty$ on d-length of causal curves σ in K.

Definition (Globally hyperbolic)

A metric spacetime (M, d, ℓ) is *globally hyperbolic* if it is non-total imprisoning and the causal diamond $J(x, y)$ is compact for each $x, y \in M$.

Definition (Timelike curve-connected; Lorentzian geodesic space)

A metric spacetime is *timelike curve-connected* iff each $x \ll y$ are connected by a timelike curve; it is a *Lorentzian geodesic space* iff each $x < y$ are connected by a causal curve σ with $L_{-\ell}(\sigma) = -\ell(\sigma(0), \sigma(1))$.

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Theorem (M. 23+: Characterizing Lorentzian length spaces "LLS")

Assuming globally hyperbolicity, a metric spacetime (M, d, ℓ) is an LLS iff it is (a) a timelike curve-connected (b) Lorentzian geodesic space; (c) ℓ is upper semicontinuous; (d) $\ell_+ = \max\{\ell, 0\}$ is continuous and (e) $I^{\pm}(x)$ both nonempty $\forall x \in M$.

- modelled on manifolds without boundary
- In such spaces, K-S showed that metric topology coincides with the order topology induced by $\ll$; this implies g.h. LLS's are independent of d!
- Burtscher & Garcia-Hevelling 21+ characterize global hyperbolicity of an LLS via existence of Cauchy time functions (and surfaces)

- Unfortunately, its not clear that all ℓ-paths are continuous!

Definition (Regular(ly localizable))

An LLS is *regular* (or *regularly localizable*) if for any $L_{-\ell}$-minimizing causal curve, $L_{-\ell}(\sigma|_{[a,b]}) = 0$ with $\sigma|_{[a,b]}$ non-constant implies $L_{-\ell}(\sigma) = 0$.

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In a globally hyperbolic regular LLS, each ℓ-path is continuous.

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Lemma (M. 23+)

In a globally hyperbolic regular LLS, each ℓ-path is continuous.

Corollary (Relating ℓ-paths to $L_{-\ell}$-extremizers)

In a globally hyperbolic regularly localizable Lorentzian length space:
(a) Every ℓ-path becomes a d-Lipschitz $L_{-\ell}$-minimizing curve after a continuous increasing (not necessarily Lipschitz) reparameterization.
(b) K-S: Conversely, every $L_{-\ell}$-minimizing curve with timelike separated endpoints becomes an ℓ-path after a similar reparameterization.

• For convenience, we deal only with metric spacetimes (M, d, ℓ) which are closed Lorentzian geodesic subsets of globally hyperbolic regular Lorentzian length spaces (= g.h.r. LLS).

Now that timelike geodesics exist:

• given a triple $x \ll y \ll z$ of timelike related events, we can compare the Lorentzian length of a bisector to that of the Minkowski triangle with the same Lorentzian sidelengths

• and similarly for generalized bisectors (i.e. ratios other than 1 : 1)

• K-S define $\text{T-Sec}(M, d, \ell) \geq 0$ if our generalized bisector is longer (and $\text{T-Sec}(M, d, \ell) \leq 0$ if it is shorter) for all such timelike triangles

• they define $\pm$ $\text{T-Sec}(M, d, \ell) \geq k \in \mathbf{R}$ analogously by comparing to timelike triangles in constant curvature Lorentzian spaces

• they also give causal sectional curvature bounds and show such bounds prevent branching of ℓ-geodesics:

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Definition (timelike nonbranching)

(M, ℓ) *timelike nonbranching* if for all $\tilde{\sigma}, \sigma \in \mathrm{TPath}^{\ell}$ with $\sigma|_{[\frac{1}{3},\frac{2}{3}]} = \tilde{\sigma}|_{[\frac{1}{3},\frac{2}{3}]}$ then $\tilde{\sigma} = \sigma$;

• Andersson-Howard '03, Alexander-Bishop '08 shows consistency of these definitions with smooth timelike sectional curvature bounds on Lorentzian manifolds

• Kunzinger-Steinbauer '22, Minguzzi-Suhr '22+ show stability of similar bounds

• Beran-Ohanyan-Rott-Solis '22+: $\text{T-Sec}(M, d, \ell) \geq 0$ and existence of a timelike line implies geometric splitting of (M, d, ℓ)

To pass from sectional to Ricci curvature / Einstein eq requires averaging:

Definition (Optimal transport distance between measures)

• Given metric spaces $(M^\pm, d^\pm)$, let $\mathcal{P}(M)$ denote the Borel probability measures on M and $\mathcal{P}_c(M)$ those with compact support.

• *Push-forward:* given $G : M^- \longrightarrow M^+$ Borel and $\mu^- \in \mathcal{P}(M^-)$, define $\mu^+ = G_\# \mu^- \in \mathcal{P}(M^+)$ by $\mu^+(B) = \mu^-(G^{-1}(B))$ for all $B \subset M^+$.

• Letting $\pi^\mp(x^-, x^+) = x^\mp$ denote the projection from $M^- \times M^+$ onto its left and right factors, set $\Gamma(\mu^-, \mu^+) = \{\gamma \in \mathcal{P}(M^- \times M^+) \mid \pi^\pm_\# \gamma = \mu^\pm\}$.

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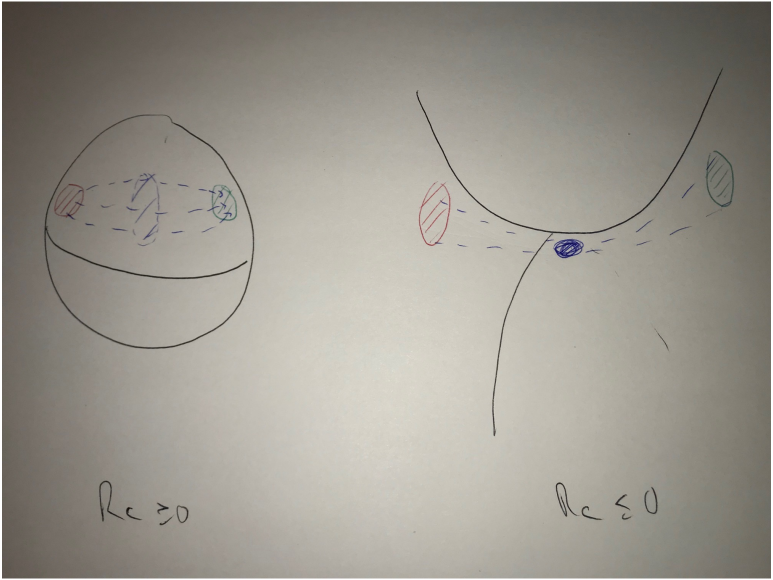
- Given $p \in [1, \infty)$ and $M = M^\pm$, the *p-Kantorovich-Rubinstein-Wasserstein distance d_p* between $\mu^\pm \in \mathcal{P}(M)$ defined by

$$d_p(\mu^-, \mu^+) := \inf_{\gamma \in \Gamma(\mu^+, \mu^-)} \left(\int_{M^2} d(x,y)^p d\gamma(x,y) \right)^{1/p} \quad (3)$$

is well-known to metrize convergence against functions growing no faster than $d(x, \cdot)^p$ provided (M, d) is *Polish* (i.e. complete and separable), in which case the inf is attained.
- If (M, d) is a geodesic space so is $(\mathcal{P}_c(M), d_p)$.


Rc ≥ 0
Rc ≤ 0

Definition (Causal and timelike measures)

In a Polish g.h.r LLS (M, d, ℓ), given $\mu, \nu \in \mathcal{P}_c(M)$ and $q \in (0, 1]$ set

$$\Gamma_{\leq}(\mu, \nu) := \{\gamma \in \Gamma(\mu, \nu) \mid \gamma[M^2_{\leq}] = 1\} = \{\textit{causal measures}\}$$

$$\Gamma_{\ll}(\mu, \nu) := \{ \quad \text{''} \quad \mid \gamma[M^2_{\ll}] = 1\} = \{\textit{timelike measures}\}$$

Lemma (Lift time-separation from events to measures)

$$\ell_q(\mu, \nu) := \max_{\gamma \in \Gamma_{\leq}(\mu,\nu)} \left(\int_{M^2} \ell(x, y)^q d\gamma(x, y) \right)^{1/q} \tag{4}$$

makes $(\mathcal{P}_c(M), \ell_q)$ into a timelike ℓ_q-path space. Not all such ℓ_q-paths are d_1-continuous;

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In a Polish g.h.r LLS (M, d, ℓ), given $\mu, \nu \in \mathcal{P}_c(M)$ and $q \in (0, 1]$ set

$$\Gamma_{\leq}(\mu,\nu) := \{\gamma \in \Gamma(\mu,\nu) \mid \gamma[M^2_{\leq}] = 1\} = \{\textit{causal measures}\}$$

$$\Gamma_{\ll}(\mu,\nu) := \{ \quad \text{''} \quad \mid \gamma[M^2_{\ll}] = 1\} = \{\textit{timelike measures}\}$$

Lemma (Lift time-separation from events to measures)

$$\ell_q(\mu,\nu) := \max_{\gamma \in \Gamma_{\leq}(\mu,\nu)} \left(\int_{M^2} \ell(x,y)^q d\gamma(x,y) \right)^{1/q} \tag{4}$$

makes $(\mathcal{P}_c(M), \ell_q)$ into a timelike ℓ_q-path space. Not all such ℓ_q-paths are d_1-continuous; one will be if (μ, ν) is timelike q-dualizable:

Definition (timelike q-dualizability)

Let $\Gamma^q = \Gamma^q(\mu,\nu)$ denote the set of maximizers. Then

- (μ,ν) are *timelike q-dualizable* if $\Gamma^q_{\ll} := \Gamma^q \cap \Gamma_{\ll}(\mu,\nu)$ is non-empty
- (μ,ν) are *strongly* timelike q-dualizable if, in addition, $\Gamma^q \subset \Gamma_{\ll}(\mu,\nu)$

Definition (Polish / proper metric-measure spacetime)

A *metric-measure spacetime* refers to a Lorentzian geodesic closed subset (M, d, ℓ) of a g.h.r. LLS, equipped with a Borel measure $m \geq 0$, finite on bounded sets. It's called *Polish* if complete and separable, and *proper* if all bounded subsets $X \subset M$ are compact.

Example (Smooth metric-measure spacetimes)

Any smooth, connected, Hausdorff, time-oriented, n-dimensional Lorentzian manifold (M^n, g) of signature $(+ - \ldots -)$ is second-countable (Ozeki-Nomizu '61) and its topology comes from a complete Riemannian metric $\tilde{g}$ (Geroch '68). With the distance $d_{\tilde{g}}$ and time-separation function ℓ_g induced by $\tilde{g}$ and g respectively, is a proper g.h.r. LLS provided it has no closed causal curves and causal diamonds $J(x, y)$ are compact. Letting $V \in C^\infty(M)$ and vol_g denote its Lorentzian volume, setting $dm = e^{-V} d\mathrm{vol}_g$ makes it a proper metric-measure spacetime. We call such spaces *smooth metric-measure spacetimes*.

Synthetic timelike Ricci bounds

Desiderata:

- consistency (with the analogous smooth bounds)
- stability (preservation under suitable limits)
- consequences (e.g. Hawking-type singularity theorem)

Definition (Entropy)

We define the relative *entropy* by

$$H(\mu \mid m) := \begin{cases} \int_M \rho \log \rho \, dm & \text{if } \mu \in \mathcal{P}_c^{ac}(M) \text{ and } \rho := \frac{d\mu}{dm}, \\ +\infty & \text{if } \mu \in \mathcal{P}_c(M) \setminus \mathcal{P}^{ac}(M). \end{cases}$$

- our sign convention is opposite to that of the physicists' entropy

Entropic weak timelike curvature-dimension conditions

Definition (TCD versus wTCD; e.g. $K = 0 = 1/N$)

For $(K, N, q) \in \mathbf{R} \times (0, \infty] \times (0, 1]$ write $(M, d, \ell, m) \in wTCD_q^e(K, N)$ if and only if every strongly timelike q-dualizable finite entropy pair $\mu_0, \mu_1 \in \mathcal{P}_c(M)$ admit a maximizer $\gamma \in \Gamma_{\ll}^q$ and corresponding ℓ_q-path $(\mu_t)_{t\in[0,1]}$ along which the entropy $t \in [0, 1] \mapsto h(t) := H(\mu_t \mid m)$ is upper-semicontinuous and distributionally solves the semiconvexity inequality

$$h''(t) \geq \frac{h'(t)^2}{N} + K\|\ell\|^2_{L^2(\gamma)}.$$

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Cavalletti-Mondino '20+ prove all limits of $TCD^e_q(K, N)$ space in a suitable (pointed measured weak) sense lie in $wTCD^e_q(K, N)$ if $N < \infty$

Pointed measured weak convergence [Cav.-Mondino 20+]

Fixing $x_j \in \mathrm{spt}\, m_j$ where m_j is a Radon measure, we say $(M_j, d_j, \ell_j, m_j, x_j) \to_{pmGL} (M_\infty, d_\infty, \ell_\infty, m_\infty, x_\infty)$ iff all $(M_j, d_j, \ell_j, m_j, x_j)$ embed d-continuously and ℓ-isometrically into a single proper g.h.r. LLS (X, d, ℓ) and after this embedding, $d(x_j, x_\infty) \to 0$ and the measures $m_j \to m_\infty$ converge weakly against continuous compactly supported test functions: i.e.

$$\lim_{j\to\infty} \int_X \phi dm_j = \int_X \phi dm_\infty \qquad \forall \phi \in C_c(X).$$

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• although the limit of $TCD_q^e(K,N)$ spaces is only $wTCD_q^e(K,N)$, Braun '22+ shows (q-essentially) timelike nonbranching $wTCD_q^e(K,N)$ spaces are $TCD_q^e(K,N)$. Hence a limit of timelike nonbranching $wTCD_q^e(K,N)$ spaces is $wTCD_q^e(K,N)$.

Positive energy ⇔ displacement convexity of entropy

DEF (N-Bakry-Emery modified Ricci tensor; cf. Erbar-Kuwada-Sturm'15)
Given $N \neq n$ and $V \in C^\infty(M^n)$ define

$$R_{ij}^{(N,V)} := R_{ij} + \nabla_i \nabla_j V - \frac{1}{N-n} (\nabla_i V)(\nabla_j V)$$

THM (M '20 Consistency) Fix $(K, N, q) \in \mathbf{R} \times (0, \infty] \times (0, 1)$ and a smooth metric-measure spacetime (M^n, g) with $dm = e^{-V} d\mathrm{vol}_g$. Then $(M, d_{\tilde{g}}, \ell_g, m) \in (w)TCD_q^e(K, N)$ if and only if either

(a) $N = n$, $V = const$ and $R_{ij} v^i v^j \geq K$ for all unit timelike $(v, x) \in TM$,

(b) $N > n$ and $R_{ij}^{(N,V)} v^i v^j \geq K$ for all unit timelike vectors $(v, x) \in TM$.

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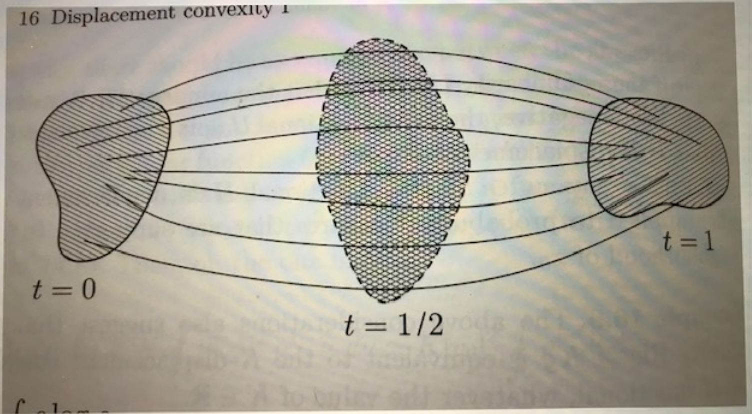
Mondino-Suhr '23 Use entropic convexity to say also when equality holds, giving a weak (but unstable) solution concept for Einstein field equation.

Akdemir-Cavalletti-Colinet-M.-Santarcangelo '21
$CD_p(K, N) \cap \{\text{nonbranching}\}$ is independent of $p > 1$

Lazy Gas Experiment (M. 94, Villani 09)



Action minimizing paths satisfy pressureless Euler equation.

Braun 23:

- $N = \infty$
- alternative definitions of $(w)TCD_q^{(*)}(K,N)$ based on convexity properties of a power-law entropy (instead of $H(\mu \mid m)$) along ℓ_q-paths

$$S_N(\mu) := -N \int_M (\frac{d\mu}{dm})^{1-\frac{1}{N}} dm$$

- equivalence of most of these various definitions to $TCD_q^e(K,N)$ assuming (q-essential) timelike nonbranching

Cavalletti-Mondino '22:

- asked for a synthetic formulation of the null energy condition (NEC)
- stronger physical motivation; more widely satisfied
- forms a key hypothesis in the Penrose singularity theorem for stellar collapse

Theorem (M' 23+)

Fix a smooth spacetime (M^n, g) with signature $(+-\cdots-)$ and symmetric 2-tensor field Q. Then

$$Q(v,v) \geq 0 \quad \forall (v,x) \in TM \text{ with } g(v,v) = 0$$

holds if and only if each compact subdomain $X \subset M^n$ admits a timelike lower bound $K = K_X$ for Q, i.e.

$$Q(v,v) \geq Kg(v,v) \quad \forall (v,x) \in TX \text{ with } g(v,v) > 0$$

Taking $Q = \mathrm{Ric}^{(N,V)}$ (or $Q_{ab} = 8\pi T_{ab}$ if Einstein holds) motivates

Definition (A synthetic null energy-dimension condition)

Given $(N,q) \in (0,\infty] \times (0,1)$, a metric-measure spacetime (M,d,ℓ,m) satisfies $wNC_q^{(e)}(N)$ if and only if each compact subset $X \subset M$ admits a bound $K = K_X \in \mathbf{R}$ such that $J(X,X) \in wTCD_q^{(e)}(K,N)$.

• in other words, the null energy condition is equivalent to a variable lower (semicontinuous) bound $k(x)$ on the timelike Ricci curvature

• Consistency with smooth (NEC) + $(n \leq N)$: follows from theorem above

• for (q-essentially) timelike nonbranching spaces $wNC_q^e(N) = NC_q^*(N)$

• Consequences: many of Cavalletti & Mondino's nice properties (timelike Bishop-Gromov and Brunn-Minkowski inequalities, needle decomposition, etc) of nonsmooth $wTCD_q^{(e)}(K, N)$ spacetimes are therefore inherited directly by $wNC_q^{(e)}(N)$ spacetimes; c.f. Braun-M. (in progress)

• (In)stability: on the other hand, any stability result appears hopeless unless we are will to assume some uniformity in j of the lower bound $k(\cdot)$ along the sequence $(M_j, d_j, \ell_j, m_j, x_j)$

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• BBCGMORS: infinitesmally Minkowski refinement of TCD_q^e, analogous to Ambrosio-Gigli-Savarè's infinitesmally Hilbertian refinement RCD of CD

• OPEN: it is natural to wonder if a Penrose singularity theorem can hold in this nonsmooth setting? (c.f. Graf '20 on $g \in C^1$ spacetimes (M^n, g), Ketterer '23+ entropic convexity derivation on $g \in C^\infty$ spacetimes)

Example (Instability displaying topology change)

Fix a slab $M := \{(x^1, \ldots, x^n) \in \mathbf{R}^n_1 \mid x^1 \in [-1, 1]\}$ of Minkowski space, with its usual metric g and $\tilde{g}$ but $dm_j(x) = e^{jg(x,x)} d\mathrm{vol}_g(x)$.
Then $(M, d_{\tilde{g}}, \ell_g, \hat{m}_j, 0) \to (M, d_{\tilde{g}}, \ell_g, m_\infty, 0)$ where $m_\infty = \frac{1}{2}(\delta_z + \delta_{-z})$ and $z = (1, 0, \ldots, 0)$. Here $d\hat{m}_j(x) = dm_j(x) / \int_M dm_j$.
Moreover $(M, d_{\tilde{g}}, \ell_g, \hat{m}_j, 0) \in NC^e_q(\infty)$ if and only if $j < \infty$

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Moreover $(M, d_{\tilde{g}}, \ell_g, \hat{m}_j, 0) \in NC^e_q(\infty)$ if and only if $j < \infty$
(and $\mathrm{spt}\, \mu_j$ is connected if and only if $j < \infty$)

Proof.

$$\mathrm{Ric}^{N,V}_x(w,w) = 0 + (-j) g(w,w) - \frac{1}{N-n} g(w,x)^2.$$

□

OPEN: Might the weak or dominant energy condition be stable?

A few references

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THANK YOU