Trading linearity for ellipticity: a nonsmooth approach to Einstein's theory of gravity and Lorentzian splitting theorems

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www.math.toronto.edu/mccann/Talk3.pdf

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What is a splitting theorem? (dimension reduction. . .)

Example (convex functions, not necessarily smooth)

If the graph of a convex function $u : \mathbf{R}^n \longrightarrow \mathbf{R}$ contains a full line, say $\frac{\partial^2 u}{\partial t^2}(t, 0, \ldots, 0) = 0$ for all $t \in \mathbf{R}$, then $u(x) = U(x_2, \ldots, x_n)$ for all $x \in \mathbf{R}^n$

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If a connected complete Ricci nonnegative Riemannian manifold (M^n, g_{ij}) contains an isometric copy of a line $(\mathbf{R}, dr^2)$, then M is a geometric product of $(\mathbf{R}, dr^2)$ with a submanifold $(\Sigma^{n-1}, h_{ij} = g_{ij}|_\Sigma)$: i.e.

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Gigli '13+: nonsmooth version for infinitesimally Hilbertian metric-measure spaces (M, d, m) satisfying curvature-dimension condition $RCD(0, N)$ à la Sturm '06, Lott & Villani '09, (... M. '94)

This talk: *Lorentzian* analogs relevant to Einstein's theory of gravity

Schematic proof of Cheeger-Gromoll splitting theorem:

Let $\gamma : \mathbf{R} \longrightarrow M^n$ be the isometrically embedded line.

Busemann '32: $b_r(x) := d(x, \gamma(r)) - d(\gamma(0), \gamma(r))$ and $\pm b^\pm := \lim_{r \to \pm\infty} b_r$

- note b_r is 1-Lipschitz and $|\nabla b_r| = 1 = |\nabla b^\pm|$ a.e.; for $r > 0$,
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- $\Sigma := \{x \in M^n \mid b(x) = 0\}$ is totally geodesic (its normal ∇b is parallel)
- along Σ, metric splits into tangent $g_{ij} dy^i dy^j$ and normal components dr^2
- $(r, y) \in \mathbf{R} \times \Sigma \mapsto \exp_y r \nabla b(y)$ is surjective hence a global isometry

General relativity: Einstein's gravity and field equation

• Einstein's gravity is formulated on smooth Lorentzian manifolds,
but often predicts such manifolds are geodesically incomplete (nonsmooth)

Gravity not a force, but rather a manifestation of curvature of spacetime
"Spacetime tells matter how to move" (along timelike/null geodesics...)

Field equation "Matter tells spacetime how to bend"

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• understanding Ricci and Einstein (à la Kip Thorne) $i, j \in \{0, 1, 2, 3\}$

• just integrate this local conservation law for $T_{ij}(x)$ to find g_{ij}...

What if matter distribution is unknown?

Can look at initial value problem (nonlinear wave equation); linearization produces gravity waves; no smoothing; singularities propagate...

Elliptic vs hyperbolic geometry

ELLIPTIC: $\mathbf{R}^n$ equipped with Euclidean norm $|v|_E := (\sum v_i^2)^{1/2}$

- $|v+w|_E \leq |v|_E + |w|_E$

HYPERBOLIC: Minkowski space $\mathbf{R}^n$ equipped with the *hyperbolic 'norm'*

$$|v|_F := \begin{cases} (v_1^2 - \sum_{i\geq 2} v_i^2)^{1/2} & v \in F := \left\{ v \in \mathbf{R}^n \mid v_1 \geq (\sum_{i\geq 2} v_i^2)^{1/2} \right\} \\ -\infty & \textit{else} \end{cases}$$

- $|v+w|_F \geq |v|_F + |w|_F$, but terribly asymmetric

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the future $F \subset \mathbf{R}^n$ is a convex cone; $v \in F$ called *causal* or *future-directed*

- v is *timelike* if $v \in F \setminus \partial F$
- v is *lightlike (or null)* if $v \in \partial F \setminus \{0\}$

(• v is *spacelike* iff $\pm v \notin F$ and *past-directed* if $-v \in F$)

- smooth curves are called *timelike (etc.)* if all tangents are timelike (etc.)

A crash course in differential geometry: action principles

Manifold M^n with symmetric nondegenerate C^k-smooth tensor $g_{ij} = g_{ji}$

RIEMANNIAN: $(g_{ij}) > 0$ defines Euclidean norm on each tangent space

- its geometry is also encoded in the (symmetric) distance function

$$d(x,y) := \inf_{\sigma(0)=x,\sigma(1)=y} \left(\int_0^1 |\dot{\sigma}_t|^q_{E_g} dt \right)^{1/q} \qquad q > 1$$

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- $(-\infty)^q := -\infty$ so $\ell(x,y) = -\infty$ unless a causal curve links x to y
- extremizers are independent of q; they are called *geodesics*
- $\ell(x,z) \geq \ell(x,y) + \ell(y,z)$ (analog of the triangle inequality d satisfies)

The Riemann curvature tensor

Given (timelike) *geodesics* $(\sigma_s)_{s\in[0,1]}$ and $(\tau_t)_{t\in[0,1]}$ with $\sigma_0 = \tau_0$ and $\dot\tau_0, \dot\sigma_0, \dot\tau_0 - \dot\sigma_0 \in F \setminus \partial F$,

$$\ell(\sigma_s, \tau_t)^2 = |t\dot\tau_0 - s\dot\sigma_0|^2_{F_g} - \frac{Sec}{6} s^2 t^2 + O((|s| + |t|)^5)$$

where sectional curvature $Sec = R(\dot\sigma_0, \dot\tau_0, \dot\sigma_0, \dot\tau_0)$ is quadratic in $\dot\sigma_0 \wedge \dot\tau_0$ and measures the leading order correction to Pythagoras

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- its trace $\mathrm{Ric}_{ik} = g^{jl}R_{ijkl}$ yields the *Ricci* tensor; $\mathrm{Ric}(v, v)$ measures the correction to Pythagoras averaged over all triangles including side v
- second trace $R = g^{ik}\mathrm{Ric}_{ik}$ yields the *scalar curvature*; in the elliptic case it gives leading order correction to the area of a sphere of radius r (and to the volume of a ball of radius r)
- $dm = e^{-V} d\mathrm{vol}_g$ where $d\mathrm{vol}_g(x) = \sqrt{|\det(g)|}d^n x$; (in Riemannian case vol_g coincides with the n-dimensional Hausdorff measure associated to d)

Energy conditions and singularity theorems

WEC (weak energy condition): $T(v, v) \geq 0$ for all future $v \in F$ (physical)
SEC (strong energy condition): $\mathrm{Ric}(v, v) \geq 0$ for all future $v \in F$ (less ")
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[Cosmological constant (dark matter): $\geq (n-1)Kg(v,v)$]

THM: Hawking '66 (big bang) singularities are generic:
SEC + mean curvature bound $H_\Sigma \geq h > 0$ on a suitable hypersurface Σ implies finite-time singularities along all timelike geodesics through Σ

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THM: Penrose '65 (stellar collapse) singularities are generic
NEC + trapped codimension-2 compact surface S + suitable noncompact hypersurface Σ imply finite-time singularity along some null geodesic

- Graf '20 $g_{ij} \in C^1$; Cavalletti-Mondino-Manini '25+ $g_{ij} \in C^0(M)$

Smooth Lorentzian splitting theorems

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• *'strong energy condition' SEC:* $g(v,v) > 0$ implies $\mathrm{Ric}(v,v) \geq 0$

• *'line'*: doubly-infinite, maximizing, timelike geodesic

• *'timelike geodesically complete'*: all (unit speed) timelike geodesics admit doubly-infinite extensions (maximizing locally but not necessarily globally)

Theorem (conjectured by Yau '82 proved by Newman '90)

Let (M^n, g_{ij}) be a SEC spacetime containing a line. If M is (a) timelike geodesically complete, then M is a geometric product of $\mathbf{R}$ *with a (Ricci nonnegative, complete) Riemannian submanifold Σ^{n-1}*

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Beem, Ehrlich, Galloway, Markvorsen '85: proved under sectional curvature bounds assuming (b) global hyperbolicity; (nonsmooth: BeOR & Solis '23)

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Nonsmooth Galloway-type theorem/new proof of Newman

- BGMOS 25+ $g_{ij} \in C^1(M^n)$ (with C^1 weights a la Case, Woolgar-Wylie)

Let $\gamma : \mathbf{R} \longrightarrow M^n$ be the isometrically embedded line and follow Busemann et al: $b_r^+(x) := -\ell(x, \gamma(r)) + \ell(\gamma(0), \gamma(r))$ and $b^\pm := \lim_{r \to \pm\infty} b_r^\pm$ where $b_r^-(x) := \ell(\gamma(r), x) - \ell(\gamma(r), \gamma(0))$

- reverse Lipschitz $b_r^\pm(y) - b_r^\pm(x) \geq \ell(x, y)$ and $|\nabla b_r^\pm|_F = 1$ a.e.; $\forall r > 0$,
- triangle inequality gives $b_r \geq b^+ \geq b^- \geq -b_{-r}$, all vanishing at $\gamma(0)$

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Let $\gamma : \mathbf{R} \longrightarrow M^n$ be the isometrically embedded line and follow Busemann et al: $b_r^+(x) := -\ell(x,\gamma(r)) + \ell(\gamma(0),\gamma(r))$ and $b^\pm := \lim_{r\to\pm\infty} b_r^\pm$ where $b_r^-(x) := \ell(\gamma(r),x) - \ell(\gamma(r),\gamma(0))$

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For $p < 1$, the operator $\square_p u := -\nabla \cdot (|\nabla u|_F^{p-2} \nabla u)$ is nonuniformly elliptic and (SEC) implies $\square_p b_r^+ \leq \frac{n-1}{\ell(\cdot,\gamma(r))}$ distributionally, i.e. $\forall\ 0 \leq \phi \in C_c^1(M)$

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$$\int_M g\left(\nabla\phi, \frac{\nabla b_r^+}{|\nabla b_r^+|_F^{2-p}}\right) d\mathrm{vol}_g \leq (n-1)\int_M \frac{\phi(\cdot)d\mathrm{vol}_g(\cdot)}{\ell(\cdot,\gamma(r))}.$$

- for the distributional limit $r \to \infty$, need $\nabla b_r^+ \longrightarrow \nabla b^+$ strongly
- need uniform ellipticity; must bound $\{\nabla b_r^+\}_{r\geq R}$ away from lightcone

Convex p-energy: trading linearity for ellipticity

Additional conditions may ensure $\ell \neq +\infty$ and extremizers exist
• complete or *proper* (boundedly compact) in the Riemannian case
• (b) *globally hyperbolic* in the Lorentzian case (i.e. compact diamonds, future F varies continuously over M, no closed future-directed curves)

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Convex Hamiltonian $H(w) = -\frac{1}{p}|w|^p_{F^*}$ (and Lagrangian $L(v) = -\frac{1}{q}|v|^q_F$) satisfy $DH = (DL)^{-1}$ if $p^{-1} + q^{-1} = 1$ (here $p < 0$ iff $0 < q < 1$)
- note $L = H^*$ jumps from 0 to $+\infty$ across future cone boundary ∂F (but H diverges continuously at the boundary of the dual cone F^*)
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Beran Braun Calisti Gigli M. Ohanyan Rott Sämann (octet): extremizers of p-Dirichlet energy $u \mapsto \int_M H(du) d\mathrm{vol}_g$ rel. to compactly supported perturbations satisfy a new nonuniformly elliptic nonlinear PDE
• trade linearity of d'Alembertian for ellipticity of p-d'Alembertian!

Nondivergence expression of (nonuniform) ellipticity

$$\Box_p b = \nabla_i \left(\frac{\partial H}{\partial w_i}\Big|_{db} \right) = H^{ij} \nabla_i \nabla_j b$$

$$H(w) = -\frac{1}{p} |w|^p_{F^*}$$

$$H^i := \frac{\partial H}{\partial w_i} = -|w|^{p-2} g^{ik} w_k$$

$$H^{ij} := \frac{\partial^2 H}{\partial w_i \partial w_j} = |w|^{p-2} \left[(2-p) g^{ik} g^{jl} \frac{w_k w_l}{|w|^2} - g^{ij} \right]$$

$$\sim |w|^{p-2} \begin{bmatrix} 2-p-1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} > 0 \quad \text{if } p < 1$$

choosing normal coordinates around $\gamma(0)$ in which $w = db$ is the time axis

Nomizu-Ozeki '61 give a complete Riemannian metric $\tilde{g}$ on (M, g).

Theorem (Eschenburg '88 ... Galloway-Horta '96)

Under (a) and/or (b), $\gamma(0)$ admits a neighbourhood X and constants R, C such that if $r \geq R$ then (i) a maximizing geodesic σ connects each $x \in X$ to $\gamma(r)$; (ii) each such geodesic satisfies $\tilde{g}(\sigma'(0), \sigma'(0)) \leq C g(\sigma'(0), \sigma'(0))$ hence $\{b_r^+\}$ is timelike and uniformly equiLipschitz on X.

- intersecting ellipsoid and hyperboloid uniformize ellipticity on X

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Lemma (BGMOS 24+: equi-semiconcavity (one derivative better))

For $g_{ij} \in C^\infty$ and some C', all $u \in \{b_r^+\}_{r \geq R}$ and $(v, x) \in TX$ satisfy

$$\lim_{t \to 0} \frac{u(\exp_x^{\tilde{g}} tv) + u(\exp_x^{\tilde{g}} -tv) - 2u(x)}{\tilde{g}(v, v)} \leq C'$$

(• p-d'Alembert comparison result then follows from smooth calculations)

- gives $db_r^+ \to db^+$ pointwise a.e., hence $|db^\pm|_{F^*} = 1$ a.e. and
- $\pm b^\pm$ are distributionally p-superharmonic $\Box_p b^+ \leq 0 \leq \Box_p b^-$

- now strong maximum principle improves $b^+ \geq b^-$ to $b^+ = b^- \in C^{1,1}(X)$
- (elliptic regularity (i.e. Schauder theory) gives $b := b^\pm \in C^\infty(X)$)

Homogeneity $2p - 2 < 0$ case of Bochner-Ohta '14, c.f. Mondino-Suhr '23: $|du|_{F^*} = 1$ and $\Box_p u = 0$ imply

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$$\begin{aligned} 0 =& \nabla_i(H^{ij}|_{du} \nabla_j (H|_{du})) - H^i \nabla_i (\nabla_j (H^j|_{du})) \\ =& H^{ij} u_{jk} H^{kl} u_{li} + R_{ij} H^i H^j \\ =& \mathrm{Tr}\left[\left(\sqrt{D^2 H} \nabla^2 u \sqrt{D^2 H}\right)^2\right] + \mathrm{Ric}(DH, DH) \end{aligned}$$

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- timelike Ricci nonnegative (i.e. SEC) gives Lorentzian $\mathrm{Hess}\, b = 0$ in X
- hence ∇b is a Killing vector field (its flow gives a local isometry on X)
- $\Sigma := \{x \in X \mid b(x) = 0\}$ is totally geodesic (its normal ∇b is parallel)
- on Σ, metric splits into tangent $g_{ij}dy^i dy^j < 0$ & normal component dr^2
- simplify Eschenburg '88, Galloway '89, Newman '90, to get from X to M

Line-adapted curves (inspired by soln to Monge's problem)

$\gamma : \mathbf{R} \longrightarrow M$ is a complete timelike line $\Leftrightarrow \ell(\gamma(s), \gamma(t)) = t - s \ \forall s < t$.

Definition

A function $\sigma : (-\epsilon, \epsilon) \longrightarrow M$ is called *γ-adapted* $\Leftrightarrow b^+(\sigma(s)) = b^-(\sigma(s))$ and $b^+(\sigma(t)) - b^+(\sigma(s)) = \ell(\sigma(s), \sigma(t)) = t - s$ for all $-\epsilon < s < t < \epsilon$.

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If σ is γ-adapted then (a) σ is a maximizing geodesic (continuous by M.24) and

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If σ is γ-adapted then (a) σ is a maximizing geodesic (continuous by M.24) and (b) any co-ray to γ starting at $\sigma(0)$ is tangent to σ, hence timelike.

Corollary (Regularity and coincidence of $b^\pm$ nearby)

It follows that $\sigma(0)$ has a neighbourhood W on which $b_r^\pm$ are equiLipschitz. Hence $b^\pm$ are Lipschitz on W, the ellipticity of $\Box_p$ becomes uniform on W, and $b^+ = b^- \in C^1(W)$ with Hessian vanishing distributionally as before (whence $b^\pm \in C^2(W)$).

The Killing field $\nabla b^+ \in C^1$. Its gradient flow

$$\frac{d\Phi_t}{dt} = \nabla b^+(\Phi_t(x))$$
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Interplay between this isometry and the co-ray construction prevents the maximal neighbourhood of U on which $b^+ = b^-$ has vanishing Hessian (and $dV(\nabla b^+) = 0$ in the weighted case) from looking like a snake that swallowed an elephant.

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Connectedness of M shows the isometric embedding is surjective.

The isometry is a posteriori C^2 (BGM.OS '25+, or apply Taylor '08 / Calabi–Hartman '70 / Myers-Steenrod '39 to the Riemannian metric $2db^+ \otimes db^+ - g$) □

THANK YOU VERY MUCH!