

Asymptotics near extinction for nonlinear fast diffusion on a bounded domain

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Archive Rational Mech. Analysis 247 (2023) 2:16 (48 pp) arXiv:2202.02769

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11 October 2023

- 1 Introduction to Nonlinear Diffusion
- 2 History and goals
- 3 Methods and results
 - The dynamical systems approach
 - Challenges
- 4 Acknowledgements

Nonlinear diffusion: basic question

Rate (and corrections) at which the nonlinear diffusion equation

$$\begin{aligned} \frac{\partial \rho}{\partial \tau} &= \frac{1}{m} \Delta(\rho^m) && \text{in } \Omega \subset \subset \mathbf{R}^n \text{ open and bounded} \\ \rho &= 0 && \text{on } (0, \infty) \times \partial\Omega \in C^\infty \\ 0 \leq \rho &= \rho_0 \in L^1(\Omega) && \text{on } \{\tau = 0\} \times \Omega \end{aligned}$$

transports heat from Ω to the sink at its boundary $\partial\Omega$?

Three regimes:

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Three regimes:

(PM) Porous medium: $m \in]1, \infty[$

(FD) Sobolev subcritical fast diffusion: $0 < m \in]\frac{n-2}{n+2}, 1[$

(FD') Sobolev supercritical fast diffusion: $m \in]-\infty, \frac{n-2}{n+2}[$

Limiting cases: linear heat equation $m = 1$

Sobolev critical diffusion $m = \frac{n-2}{n+2}$

How does this work for the linear heat equation $m = 1$?

Recall: separation of variables yields

$$\rho(\tau, \mathbf{y}) = \sum_{i=1}^{\infty} c_i e^{-\lambda_i \tau} \phi_i(\mathbf{y})$$

where

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots,$$

and $c_i = \langle \rho_0, \phi_i \rangle_{L^2}$ where $\{\phi_i\}_{i=1}^{\infty} \subset H_0^1(\Omega)$ for

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$$H_0^1(\Omega) = \{\phi \in L^2(\Omega) \mid D\phi \in L^2(\Omega) \text{ and } \phi = 0 \text{ on } \partial\Omega\}$$

solve

$$-\Delta \phi_i = \lambda_i \phi_i \quad \text{on } \Omega$$

and form an orthonormal basis for $L^2(\Omega)$

Do the nonlinear dynamics admit a similar description?

$$\begin{cases} \rho(0, y) = \rho_0(y) \\ \frac{\partial \rho}{\partial \tau} = \frac{1}{m} \Delta(\rho^m) \end{cases}$$

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POROUS MEDIUM REGIME ($m > 1$)

- fluid in rock; population spreading; temperature dependent conductivity
- rate of diffusion ρ^{m-1} varies **directly** with density ρ of diffusing material
- compactly supported ρ_0 remains compactly supported at $\tau > 0$

Motivation: dissipative fluids

$$\left(\frac{\partial}{\partial t} + u \cdot \nabla\right)(\rho u) = -\nabla P(\rho) - bu \quad (1)$$

- if drag negligible ($b \ll 1$), (1) couples with continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \quad (2)$$

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- if drag dominates ($b \gg 1$), neglect **inertial** terms in (1); then (2) yields

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- polytropic equation of state $P(\rho) = \frac{b}{m-1} \rho^m$ gives nonlinear diffusion (3)

Subcritical fast diffusion regime ($\frac{n-2}{n+2} < m < 1$)

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$$v(t, x)^p = \frac{\rho(\tau, x)}{\frac{1-m}{m} (T - \tau)^{\frac{1}{1-m}}} \quad \text{and} \quad t = -\frac{1-m}{m} \log \left| 1 - \frac{\tau}{T} \right|$$

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$$\frac{\partial}{\partial t} \left(\frac{v^p}{p} \right) = \Delta v + v^p = \frac{\delta E}{\delta v} \quad \text{in } (t, x) \in (0, \infty) \times \Omega$$

where

$$E(v) := \int_{\Omega} \left[\frac{1}{2} |\nabla v|^2 - \frac{1}{p+1} v^{p+1} \right] dx$$

is a **Lyapunov** function for the rescaled dynamics.

- **Berger '77**: $\frac{\delta E}{\delta v} = 0$ has positive solutions in $H_0^1(\Omega)$ 'ground states'
- **Berryman-Holland '80**: $v(t) \rightarrow$ ground state as $t \rightarrow \infty$ along a **subsequence**; conjectured limit unique & higher-order asymptotics
- **Brezis-Nirenberg '83**: ground states **non-unique** on certain domains

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- Brezis-Nirenberg '83: ground states **non-unique** on certain domains
- Feireisl-Simondon '00: $\lim_{t \rightarrow \infty} v(t) = V$, but depends on $v(0)$
- Bonforte-Grillo-Vazquez '12: $\left\| \frac{v(t)}{V} - 1 \right\|_{\infty} \rightarrow 0$ (rate if $m \sim 1$)
- Jin-Xiong '20+: $\left\| \frac{v(t)}{V} - 1 \right\|_{\infty} \leq \frac{C}{t^{\sigma}}$ for some $C, \sigma > 0$ if $m \in [\frac{n-2}{n+2}, 1[$
- Bonforte-Figalli '21: $\left\| \frac{v(t)}{V} - 1 \right\|_{\infty} \leq C e^{-\lambda t}$, where the **spectral gap** $\lambda > 0$ for an open $C^{2,\alpha}$ dense set of domains Ω , including the ball
- Akagi '21+ energetic (rather than entropic) proof

Linearization

relative error

$$h(t) := \frac{v(t)}{V} - 1$$

satisfies

$$\frac{\partial h}{\partial t} + L_V h = N(h) = M_V(h)$$

where

$$\begin{aligned} L_V h &= -\frac{1}{V} \Delta(hV) - ph \\ &= -V^{-p-1} \nabla \cdot (V^2 \nabla h) - (p-1)h \\ &\geq (1-p)h \end{aligned}$$

$$N(h) = (1+h)^p - 1 - ph - ((1+h)^p - 1)^p \frac{\partial h}{\partial t}$$

$$M_V(h) = \frac{1}{(1+h)^{p-1}} [(1+h)^p - 1 - ph + ((1+h)^p - 1)L_V h]$$

Diagonalization (Bonforte-Figalli '21)

$$\|f\|_{L_r^q} := \left(\int_{\Omega} |f(x)|^q V(x)^r dx \right)^{1/q}$$
$$\langle f, g \rangle_r := \langle f, g \rangle_{L_r^2} = \int_{\Omega} fg V(x)^r dx$$

implies L_V is self-adjoint on L_{p+1}^2 and has a complete basis of eigenvectors, which are critical points for the restriction of the weighted Dirichlet energy

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$$Q_V(\phi) := \|D\phi\|_{L_2^2}^2 = \int_{\Omega} |D\phi|^2 V^2 dx$$

to the L_{p+1}^2 unit-sphere, with the boundary trace of ϕV vanishing.

Proof: $\tilde{L} := V \circ (L_V + pI) \circ V^{-1} \geq I$ has compact inverse on L_{p-1}^2 ... \square

Denote the eigenvalues by

$$\lambda_{-l} < \lambda_{-l+1} \leq \cdots \leq \lambda_{-3} \leq \lambda_{-2} \leq \cdots \leq \lambda_{K-1} \leq 0 < \lambda_K \leq \lambda_{K+1} \leq \cdots$$

where $l \geq 1$ counts the number of **unstable modes**, $K \geq 0$ the **zero modes** (if any), and $\lambda_K > 0$ is the first positive eigenvalue or '**spectral gap**'

Note $L_V + (p-1) \geq 0$ and $L_V \mathbf{1} = (1-p)\mathbf{1}$ imply $\lambda_{-l} = 1-p$ and **simple**; this corresponds to time translation symmetry in the original variables.

Bonforte-Figalli '21's exponential convergence rate $\lambda = \lambda_K$ follows from the fact that the unstable modes are suppressed (**Feireisl-Simondon '00**), while $C^{2,\alpha}$ -generic domains admit no zero modes (**Saut-Teman '79**).

- Can we (a) **close the gap** between **Bonforte-Figalli '21**'s **exponential** and **Jin-Xiong '20**'s **algebraic** rate of convergence, and/or (b) **access higher asymptotics** conjectured (for $n=1$) by **Berryman-Holland '80**?
- (c) When do zero modes spoil exponential convergence?

First dichotomy

Theorem (Choi-M.-Seis)

Fix $\Omega \subset \subset \mathbf{R}^n$ bounded with $\partial\Omega \in C^\infty$ and $0 < m \in]\frac{n-2}{n+2}, 1[$.

If $0 \leq v \in L^\infty([0, \infty[\times \Omega)$ solves dynamics and $h(t) := \frac{v(t)}{V} - 1 \rightarrow 0$ uniformly, there exist $\epsilon, C(p, V)$ and $\lambda \geq \lambda_K$ such that $\|h\|_{L^\infty(\mathbf{R}_+ \times \Omega)} \leq \epsilon$ implies either

$$C\|h(t)\|_{L^\infty} \geq \|h(t)\|_{L^2_{p+1}} \geq \frac{1}{Ct} \quad \forall t \gg 1 \quad (4)$$

or $\frac{1}{C}\|h(t)\|_{L^2_{p+1}} \leq \|h(t)\|_{L^\infty} \leq Ce^{-\lambda t}\|h(0)\|_{L^2_{p+1}} \quad \forall t \geq 1. \quad (5)$

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If (5) holds and $2\lambda \in]\lambda_J, \lambda_{J+1}]$, then there exist $c_i = c_i(h(0)) \in \mathbf{R}$ and $\tilde{C} = \tilde{C}(V, p, \lambda, C)$ such that the eigenfunctions $L_V \phi_i = \lambda_i \phi_i \in L^2_{p+1}(\Omega)$ yield

$$\left\| h(t) - \sum_{i=K}^J c_i e^{-\lambda_i t} \phi_i \right\|_{L^2_{p+1}} \leq \tilde{C} \|h(0)\|_{L^2_{p+1}} t e^{-2\lambda t}$$

Let $S \subset H_0^1(\Omega)$ denote the set of fixed points $V \geq 0$ of the rescaled dynamics. Then $S \subset C^{3,\alpha}(\Omega)$ and $V, W \in S$ implies $V/W \in L^\infty$. Topologize S using the relatively uniform 'balls'

$$B_r(V) := \{W \in S \mid \|\frac{W}{V} - 1\|_\infty < r\}$$

as a base. Call $V \in S$ an *ordinary limit* iff S forms a *manifold* of dimension $\dim(S) = K := \dim(\text{Ker}L_V)$ near V , which the error relative to V embeds *differentiably* into $L_{p+1}^2(\Omega)$.

Theorem (Second dichotomy)

Under the hypotheses of the preceding theorem, convergence is exponentially fast if V is an ordinary limit.

Remark: All **tangent vectors** to the embedding of S at V lie in $\text{Ker}L_V$. Conversely, if V is an ordinary limit, then each $u \in \text{Ker}L_V$ is **tangent** to the embedding of S . In the latter case the kernel is said to be *integrable*, a notion exploited by [Allard-Almgren '81](#) and [Simon '85](#) for related purposes in the context of minimal surfaces and geometric evolution equations.

We expect ordinary limits to be in some sense **generic** in S .

Some history for $\Omega = \mathbf{R}^n$ with $m_q = 1 - \frac{2}{n+q}$

For compactly supported non-negative initial data with $\|\rho_0\|_1 = \|\tilde{\rho}_0\|_1$
Friedman-Kamin '80 $m > m_0 \Rightarrow \|\rho(\tau, \cdot) - \tilde{\rho}(\tau, \cdot)\|_{L^1} = o(1)$ as $\tau \rightarrow \infty$

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Kim-McCann ('05, $m \in]m_0, m_2]$, $\langle x \rangle_{\rho - \tilde{\rho}} = 0$) $= O(\tau^{-1})$ as $\tau \rightarrow \infty$

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Angenent '88 invariant manifolds $m \geq 1 = n$; Koch '99, Seis '15+ $n \geq 1$

A (finite dimensional) dynamical systems approach

$$x'(t) = -F(x(t)) \in \mathbf{R}^n \quad \text{with} \quad x(0) = x_0$$

LINEARIZE around fixed point $F(x_\infty) = 0$ to get:

$$(x(t) - x_\infty)' = -DF(x_\infty)(x(t) - x_\infty)' + O(x(t) - x_\infty)^2$$

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If $\sigma(DF(x_\infty)) = \{0 < \lambda_1 \leq \dots \leq \lambda_n\}$ with eigenvectors $\hat{\phi}_i$, as $t \rightarrow \infty$ expect

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- to **simplify**, only strive for asymptotics to order $O(e^{-2\lambda_1 t})$
- **differentiability** of $F(x_0)$ or $x(t) = X(t, x_0)$ wrt $x_0 \in \mathbf{R}^n$ was crucial

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using $|N(v(t))| \leq |v(t)| \|\frac{\partial}{\partial t} v(t)\|_{L^\infty}$ to reduce to one of the model cases

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$$\dot{x}_i = -\lambda_i x_i + N_i(x), \quad x = (x_1, \dots, x_n) \in \mathbf{R}^n$$

where $N(x) \leq k|x|^2$ if $|x| \leq \epsilon^2$, and $\lambda_i \geq \lambda > 0$. In this case Gronwall implies $f(t) = |x(t)|^2$ satisfies

$$f(T) \leq f(0) \exp[-2(\lambda - k\epsilon)T],$$

and then, if $f(t) \leq C^2 \exp[-2\Lambda t]$ for some $\Lambda > 0$, C and all $t \geq 0$,

$$|x(T)| \leq |x(0)| \exp[-\lambda T + \frac{C}{2\Lambda}]$$

Lemma (K. Choi-Haslhofer-Hershkovits '18+)

Let $X(s)$, $Y(s)$, and $Z(s)$ be non-negative AC functions on $[0, \infty)$ satisfying

$$\frac{dX}{ds} - X \geq -\epsilon(Y + Z),$$

$$\left| \frac{dY}{ds} \right| \leq \epsilon(X + Y + Z), \text{ and}$$

$$\frac{dZ}{ds} + Z \leq \epsilon(X + Y)$$

for each $\epsilon \in (0, \frac{1}{100})$ and a.e. $s \in [s_0(\epsilon), \infty)$.

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for each $\epsilon \in (0, \frac{1}{100})$ and a.e. $s \in [s_0(\epsilon), \infty)$. If $\lim_{s \rightarrow \infty} (X + Y + Z)(s) = 0$ then $X \leq 2\epsilon(Y + Z)$ for $s \geq s_0(\epsilon)$ and either

$$X(s) + Z(s) = o(Y(s)) \text{ as } s \rightarrow \infty \quad (6)$$

or

$$X(s) + Y(s) \leq 100\epsilon Z(s) \text{ for } s \geq s_0(\epsilon). \quad (7)$$

Quadratic Hilbert-space estimate for the nonlinearity:

First dichotomy: apply the lemma to $X(t)$, $Y(t)$ and $Z(t)$ defined as the Hilbert norm $L_{p+1}^2 := L^2(V^{p+1})$ of the orthogonal projection of $h(t)$ onto the **unstable**, **neutral**, and stable modes respectively. To absorb the nonlinearity into the ϵ corrections, apply the following theorem with $t = 1$.

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Theorem (Spatially uniform control of time derivatives)

Let $k \in \{0, 1, 2, \dots\}$ and $t > 0$ fixed. Then if $\|h\|_{L^\infty} \leq \epsilon$ with ϵ sufficiently small, there exists a constant $C = C(t, k, m, V)$ such that

$$\|\partial_t^k h(t)\|_{L^\infty} \leq C \|h_0\|_{L_{p+1}^2}.$$

Proof: degenerate parabolic smoothing, with delicate control near the boundary of Ω where $V(x) \sim d_{\partial\Omega}(x)$. □

c.f. [Jin-Xiong '19+](#)

Among other ingredients, second dichotomy relies on

Lemma (K. Choi-Sun '20+)

Suppose $X(s)$, $Y(s)$, and $Z(s)$ are non-negative absolutely continuous functions on some interval $[-L, L]$ such that $0 < X + Y + Z < \eta$ for some $\eta > 0$. Suppose that there exist two constants $\sigma > 0$ and $\Lambda > 0$ such that

$$\begin{aligned}\frac{dX}{ds} - \Lambda X &\geq -\sigma(Y + Z), \\ \left| \frac{dY}{ds} \right| &\leq \sigma(X + Y + Z), \\ \frac{dZ}{ds} + \Lambda Z &\leq \sigma(X + Y),\end{aligned}$$

for any $s \in [-L, L]$. Then there exists $\sigma_0 = \sigma_0(\Lambda)$ such that if $0 < \sigma < \sigma_0$ it holds

$$X + Z \leq \frac{8\sigma}{\Lambda} Y + 4\eta e^{-\frac{\Lambda L}{4}} \text{ for any } s \in [-L/2, L/2].$$

Thank you!