

Trading linearity for ellipticity: a nonsmooth approach to Einstein's theory of gravity and Lorentzian splitting theorems

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www.math.toronto.edu/mccann/Talk3.pdf

23 September 2024

What is a splitting theorem? (dimension reduction...)

Example (convex functions, not necessarily smooth)

If the graph of a **convex** function $u : \mathbf{R}^n \rightarrow \mathbf{R}$ contains a full line, say $\frac{\partial^2 u}{\partial t^2}(t, 0, \dots, 0) = 0$ for all $t \in \mathbf{R}$, then $u(x) = U(x_2, \dots, x_n)$ for all $x \in \mathbf{R}^n$

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Example (smooth Riemannian manifolds; Cheeger-Gromoll '71)

If a connected **complete Ricci nonnegative** Riemannian manifold (M^n, g_{ij}) contains an isometric copy of a line (\mathbf{R}, dr^2) , then M is a geometric product of (\mathbf{R}, dr^2) with a submanifold $(\Sigma^{n-1}, h_{ij} = g_{ij}|_{\Sigma})$: i.e.

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Gigli '13+: nonsmooth version for **infinitesimally Hilbertian**

metric-measure spaces (M, d, m) satisfying **curvature-dimension** condition **$RCD(0, N)$** à la Sturm '06, Lott & **Villani** '09, (... M. '94)

This talk: **Lorentzian** analogs relevant to Einstein's theory of gravity

Schematic proof of Cheeger-Gromoll splitting theorem:

Let $\gamma : \mathbf{R} \rightarrow M^n$ be the isometrically embedded line.

Busemann '32: $b_r(x) := d(x, \gamma(r)) - d(\gamma(0), \gamma(r))$ and $\pm b^\pm := \lim_{r \rightarrow \pm\infty} b_r$

- note b_r is 1-Lipschitz and $|\nabla b_r| = 1 = |\nabla b^\pm|$ a.e.; for $r > 0$,
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- hence ∇b is a '**Killing**' vector field (its **flow gives a local isometry**)
- $\Sigma := \{x \in M^n \mid b(x) = 0\}$ is totally geodesic (its **normal ∇b is parallel**)
- along Σ , metric **splits** into **tangent $g_{ij} dy^i dy^j$** and **normal components dr^2**
- $(r, y) \in \mathbf{R} \times \Sigma \mapsto \exp_y r \nabla b(y)$ is **surjective** hence a **global isometry**

General relativity: Einstein's gravity and field equation

- Einstein's gravity is formulated on smooth Lorentzian manifolds, but often predicts such manifolds are geodesically incomplete (**nonsmooth**)

Gravity not a force, but rather a manifestation of curvature of spacetime

“Spacetime tells matter how to move” (along timelike/null geodesics...)

Field equation “Matter tells spacetime how to bend”

geometry = *physics*

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- just integrate this local conservation law for $T_{ij}(x)$ to find $g_{ij} \dots$

What if matter distribution is unknown?

Can look at initial value problem (nonlinear **wave** equation); linearization produces gravity waves; **no smoothing; singularities propagate**...

Elliptic vs hyperbolic geometry

ELLIPTIC: \mathbf{R}^n equipped with Euclidean norm $|v|_E := (\sum v_i^2)^{1/2}$

- $|v + w|_E \leq |v|_E + |w|_E$

HYPERBOLIC: Minkowski space \mathbf{R}^n equipped with the *hyperbolic 'norm'*

$$|v|_F := \begin{cases} (v_1^2 - \sum_{i \geq 2} v_i^2)^{1/2} & v \in F := \left\{ v \in \mathbf{R}^n \mid v_1 \geq (\sum_{i \geq 2} v_i^2)^{1/2} \right\} \\ -\infty & \text{else} \end{cases}$$

- $|v + w|_F \geq |v|_F + |w|_F$, but terribly asymmetric

the *future* $F \subset \mathbf{R}^n$ is a convex cone; $v \in F$ called *causal* or *future-directed*

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the *future* $F \subset \mathbf{R}^n$ is a convex cone; $v \in F$ called *causal* or *future-directed*

- v is *timelike* if $v \in F \setminus \partial F$

- v is *lightlike (or null)* if $v \in \partial F \setminus \{0\}$

(• v is *spacelike* iff $\pm v \notin F$ and *past-directed* if $-v \in F$)

- smooth *curves* are called *timelike (etc.)* if all tangents are timelike (etc.)

A crash course in differential geometry: action principles

Manifold M^n with symmetric nondegenerate C^k -smooth tensor $g_{ij} = g_{ji}$

RIEMANNIAN: $(g_{ij}) > 0$ defines Euclidean norm on each tangent space

- its geometry is also encoded in the (symmetric) **distance** function

$$d(x, y)^q := \inf_{\sigma(0)=x, \sigma(1)=y} \int_0^1 |\dot{\sigma}_t|_{E_g}^q dt \quad q > 1$$

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- $(-\infty)^q := -\infty$ so $\ell(x, y) = -\infty$ unless a causal curve links x to y
- extremizers are independent of q ; they are called **geodesics**
- $\ell(x, z) \geq \ell(x, y) + \ell(y, z)$ (analog of the **triangle inequality** d satisfies)

The Riemann curvature tensor

Given (timelike) *geodesics* $(\sigma_s)_{s \in [0,1]}$ and $(\tau_t)_{t \in [0,1]}$ with $\sigma_0 = \tau_0$ and $\dot{\tau}_0, \dot{\sigma}_0, \dot{\tau}_0 - \dot{\sigma}_0 \in F \setminus \partial F$,

$$\ell(\sigma_s, \tau_t)^2 = |t\dot{\tau}_0 - s\dot{\sigma}_0|_{F_g}^2 - \frac{\text{Sec}}{6} s^2 t^2 + O((|s| + |t|)^5)$$

where sectional curvature $\text{Sec} = R(\dot{\sigma}_0, \dot{\tau}_0, \dot{\sigma}_0, \dot{\tau}_0)$ is quadratic in $\dot{\sigma}_0 \wedge \dot{\tau}_0$ and measures the leading order correction to Pythagoras

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- its trace $\text{Ric}_{ik} = g^{jl} R_{ijkl}$ yields the *Ricci* tensor; $\text{Ric}(v, v)$ measures the correction to Pythagoras averaged over all triangles including side v
- second trace $R = g^{ik} \text{Ric}_{ik}$ yields the *scalar curvature*; in the elliptic case it gives leading order correction to the area of a sphere of radius r (and to the volume of a ball of radius r)
- $d\text{vol}_g(x) = \sqrt{|\det(g)|} d^n x$ in coordinates; (in the Riemannian case it coincides with the n -dimensional Hausdorff measure associated to d)

Energy conditions and singularity theorems

- WEC** (weak energy condition): $T(v, v) \geq 0$ for all **future** $v \in F$ (**physical**)
- SEC** (strong energy condition): $\text{Ric}(v, v) \geq 0$ for all **future** $v \in F$ (**less** ")
- NEC** (null energy condition): " ≥ 0 for all **lightlike** $v \in \partial F$

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[Cosmological constant (dark matter): $\geq (n-1)Kg(v, v)$]

THM: Hawking '66 (big bang) singularities are generic:

SEC + mean curvature bound $H_\Sigma \geq h > 0$ on a suitable hypersurface Σ
implies finite-time singularities along all timelike geodesics through Σ

Cavalletti-Mondino '20+: also in $TCD(0, N)$ metric-measure spacetimes

THM: Penrose '65 (stellar collapse) singularities are generic

NEC + trapped codimension-2 compact surface S + suitable noncompact hypersurface Σ imply finite-time singularity along some null geodesic

Graf '20 holds for $g_{ij} \in C^1$;

Open: version for $TCD(0, N)$ metric-measure spacetimes?

Smooth Lorentzian splitting theorems

- '*spacetime*': a connected Lorentzian manifold (M^n, g_{ij}) which admits a continuous choice of F_g (distinguishing future from past).
- '*strong energy condition*' *SEC*: $g(v, v) > 0$ implies $\text{Ric}(v, v) \geq 0$
- '*line*': doubly-infinite, maximizing, timelike geodesic
- '*timelike geodesically complete*': all (unit speed) timelike geodesics admit doubly-infinite extensions (maximizing locally but not necessarily globally)

Theorem (conjectured by Yau '82; proved by Newman '90)

Let (M^n, g_{ij}) be a SEC spacetime containing a line. If M is (a) *timelike geodesically complete*, then M is a geometric product of \mathbf{R} with a (Ricci nonnegative, complete) Riemannian submanifold Σ^{n-1}

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Galloway '84: proved assuming **compact** Cauchy surfaces exist

Beem, Ehrlich, Galloway, Markvorsen '85: proved under sectional curvature bounds assuming (b) **global hyperbolicity**; (**nonsmooth**: BeOR & Solis '23)

Eschenburg '88: proved under (a) + (b)

Galloway '89: proved under (b) **without** (a) using Bartnik '88

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Elliptic proof of the Newman (and Galloway) theorems

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Theorem (BBCGMORS 24+ ‘**nonsmooth** p -d’Alembert comparison’)

For $p < 1$, the operator $\square_p u := -\nabla \cdot (|\nabla u|_F^{p-2} \nabla u)$ is **nonuniformly elliptic** and (SEC) implies $\square_p b_r^+ \leq \frac{n-1}{\ell(\cdot, \gamma(r))}$ distributionally, i.e. $\forall 0 \leq \phi \in C_c^1(M)$

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$$\int_M g \left(\nabla \phi, \frac{\nabla b_r^+}{|\nabla b_r^+|_F^{2-p}} \right) d\text{vol}_g \leq (n-1) \int_M \frac{\phi(\cdot) d\text{vol}_g(\cdot)}{\ell(\cdot, \gamma(r))}.$$

- for the distributional limit $r \rightarrow \infty$, need $\nabla b_r^+ \rightarrow \nabla b^+$ strongly
- need **uniform ellipticity**; must bound $\{\nabla b_r^+\}_{r \geq R}$ away from lightcone

Convex p -energy: trading linearity for ellipticity

Additional conditions may ensure $\ell \neq +\infty$ and extremizers exist

- complete or *proper* (boundedly compact) in the Riemannian case
- (b) *globally hyperbolic* in the Lorentzian case (i.e. compact diamonds, future F varies continuously over M , no closed future-directed curves)

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Convex Hamiltonian $H(w) = -\frac{1}{p}|w|_{F^*}^p$ (and **Lagrangian** $L(v) = -\frac{1}{q}|v|_F^q$)

satisfy $DH = (DL)^{-1}$ if $p^{-1} + q^{-1} = 1$ (here $p < 0$ iff $0 < q < 1$)

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Beran Braun Calisti Gigli M. Ohanian Rott Sämann (octet):

extremizers of p -Dirichlet energy $u \mapsto \int_M H(du) d\text{vol}_g$ rel. to compactly supported perturbations satisfy a **new nonuniformly elliptic nonlinear** PDE

- trade linearity of d'Alembertian for ellipticity of **p -d'Alembertian**!

Nondivergence expression of (nonuniform) ellipticity

$$\square_p b = \nabla_i \left(\frac{\partial H}{\partial w_i} \Big|_{db} \right) = H^{ij} \nabla_i \nabla_j b$$

$$H(w) = -\frac{1}{p} |w|_{F^*}^p$$

$$H^i := \frac{\partial H}{\partial w_i} = -|w|^{p-2} g^{ik} w_k$$

$$H^{ij} := \frac{\partial^2 H}{\partial w_i \partial w_j} = |w|^{p-2} \left[(2-p) g^{ik} g^{jl} \frac{w_k w_l}{|w|^2} - g^{ij} \right]$$

$$\sim |w|^{p-2} \begin{bmatrix} 2-p-1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} > 0 \quad \text{if } p < 1$$

choosing normal coordinates around $\gamma(0)$ in which $w = db$ is the time axis

Nomizu-Ozeki '61 give a complete Riemannian metric \tilde{g} on (M, g) .

Theorem (Eschenburg '88 ... Galloway-Horta '96)

Under (a) and/or (b), $\gamma(0)$ admits a neighbourhood X and constants R, C such that if $r \geq R$ then (i) a maximizing geodesic σ connects each $x \in X$ to $\gamma(r)$; (ii) each such geodesic satisfies $\tilde{g}(\sigma'(0), \sigma'(0)) \leq Cg(\sigma'(0), \sigma'(0))$ hence $\{b_r^+\}$ is timelike and uniformly *equiLipschitz* on X .

- intersecting ellipsoid and hyperboloid *uniformize ellipticity* on X

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Lemma (BGMOS quintet: *equi-semiconcavity* (one derivative better))

For another constant C' , all $u \in \{b_r^+\}_{r \geq R}$ and $(v, x) \in TX$ satisfy

$$\lim_{t \rightarrow 0} \frac{u(\exp_x^{\tilde{g}} tv) + u(\exp_x^{\tilde{g}} -tv) - 2u(x)}{\tilde{g}(v, v)} \leq C'$$

(• p -d'Alembert comparison result then follows from *smooth* calculations)

- gives $db_r^+ \rightarrow db^+$ pointwise a.e., hence $|db^\pm|_{F^*} = 1$ a.e. and
- $\pm b^\pm$ are distributionally p -superharmonic $\square_p b^+ \leq 0 \leq \square_p b^-$

- now strong maximum principle improves $b^+ \geq b^-$ to $b^+ = b^- \in C^{1,1}(X)$
- (elliptic regularity (i.e. [Schauder](#) theory) gives $b := b^\pm \in C^\infty(X)$)

Homogeneity $2p - 2 < 0$ variant on Bochner [Ohta '14](#), [Mondino-Suhr '23](#):
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$$\begin{aligned}
 0 &= \nabla_i (H^{ij}|_{du} \nabla_j (H|_{du})) - H^i \nabla_i (\nabla_j (H^j|_{du})) \\
 &= H^{ij} u_{jk} H^{kl} u_{li} + R_{ij} H^i H^j \\
 &= \text{Tr} \left[\left(\sqrt{D^2 H} \nabla^2 u \sqrt{D^2 H} \right)^2 \right] + \text{Ric}(DH, DH)
 \end{aligned}$$

- timelike Ricci nonnegative (i.e. [SEC](#)) gives Lorentzian $\text{Hess } b = 0$ in X

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- timelike Ricci nonnegative (i.e. **SEC**) gives Lorentzian **Hess** $b = 0$ in X
- hence ∇b is a **Killing** vector field (its **flow gives a local isometry** on X)
- $\Sigma := \{x \in X \mid b(x) = 0\}$ is totally geodesic (its **normal** ∇b is **parallel**)
- on Σ , metric **splits** into **tangent** $g_{ij} dy^i dy^j < 0$ & **normal** component dr^2
- **simplify** **Eschenburg '88, Galloway '89, Newman '90**, to get from X to M

THANK YOU VERY MUCH!