MAT 1060H1F

Assignment 1

Prof. McCann — Due: Wednesday Oct. 4

Read Evans Appendix E. We have covered Chapeter 1, and 2.1–2.2 (except for 2.2.4 which you will need to refer to for # 2.8 below, and the proof of analyticity). If you like to read ahead I expect to get through Chapter 2.3 next week and 2.4 the week after.

To be handed in: Evans # 1.3, 2.4, 2.5, 2.8 and

1. Let given $U \subset\subset \mathbf{R}^n$ and $g \in C(\partial U)$ let $W = \{u \in C^2(\bar{U}) \mid u = g \text{ on } \partial U\}$. Let

$$A(u) := \int_U \sqrt{1 + |Du(x)|^2}\, dx$$

denote the n-dimensional area of the graph of u, and let $m = \inf_{w \in W} A(w)$ the minimum area of all graphs $w \in W$ satisfying the boundary conditions.

(a) Find a second order nonlinear partial differential equation satisfied by any area minimizing graph $u \in W$ such that $A(u) = m$. (HINT: Use the calculus of variations. An equation derived by finding critical points of a functional is called an *Euler-Lagrange* equation.) This particular PDE is also called the *minimal surface equation*; it represents the equilibrium shape a soap film whose boundary lies on a wire given by the graph of the function $g \in C(\partial U)$.

(b) Show $A : W \longrightarrow \mathbf{R}$ is strictly convex, meaning $w_0, w_1 \in W$ and $s \in]0, 1[$ imply $A((1-s)w_0 + sw_1) < (1-s)A(w_0) + sA(w_1)$ unless $w_0 = w_1$.

(c) Prove at most one function $u \in W$ satisfies the minimal surface equation. (HINT: First prove that the derivative of a convex function $a : \mathbf{R} \longrightarrow \mathbf{R}$ vanishes *only* at the minimum of a.)

2. The attached exercises #1-3 pp 116–117 from Adams and Guillemin "Measure Theory and Probability". I will grade it only for effort since it requires some elementary background on measure theory and integration which not all of you may have, but it provides a lot of insight into Green's functions. Try it in any case; you may want to spend some time acquiring enough background to do it on your own. Adams & Guillemin's book is a good place to learn. I have requested this book and Evans' to be recalled and placed on course reserve in the math library.