MAT 1060H1F

Assignment 4

Prof. McCann Due: in class 9 AM Thursday Nov. 23

Recall that I will be out of the country Nov 13–17, so office hours and lectures that week are cancelled; our first meetings after that will be 14h10-16h00 Wednesday Nov. 22 in BA 6180 (usual time and location, the full 2 hours). On Thursday Nov. 23 we will meet 09h10 - 10h00 in BA 6183 instead of the afternoon. I will hold office hours 18h10-18h50 on Nov 22 instead of on Nov. 23 (remind me to confirm this time in class on Nov. 22). For the remainder of semester we revert to normal schedule and rooms (BA 6180 on Wednesdays, BA 6183 Thursdays).

Read Evans sections 4.4.1 and 4.5.2. We will not have time to cover the rest of Chapter 4 in class, so you may want to take the week of Nov 13-17 to familiarize yourself with the material there. I plan to begin Chapter 5 on Nov 22.

To be handed in: Evans # 3.11, 3.12, 3.13, 3.14 and 4.3; plus

6. VISCOUS FINGERING: Imagine a layer of heavy fluid of density ρ_0 floating on top of a light fluid of density $\rho_1 < \rho_0$. Any slight perturbation of the flat interface between the two fluids will destabilize the situation, causing fingers of heavy fluid to penetrate into and displace the light fluid until eventually all of the heavy fluid has settled to the bottom, and all of the light fluid has risen to the top. The size of these fingers is determined by surface tension; in its absence the rate of growth of a finger is inversely proportionate to its thickness, leading to an ill-posed problem featuring complicated labyrithine geometry made up of zillions of tiny fingers and filaments. Suppose we are interested not in the details of this geometry, but only in the time T which it takes for the fluids to exchange places, and in the density profile $\rho(z,t) = (1-u(z,t))\rho_0 + u(z,t)\rho_1$ at each height $z \in [-1,1]$ in the *mixing layer* at time $0 < t < T$, and the overall thickness of this layer $\{z \in [-1,1] \mid 0 < u(z,t) < 1\}$.

In a seminal work (Comm. Pure Appl. Math. **52** (1999) 873–915), F. Otto proposed that the evolution of $u(z,t)$ would be governed by the conservation law

$$0 = \frac{\partial u}{\partial t} + \frac{\partial}{\partial z}[u(1-u)]$$

subject to the initial conditions

$$u(z,0) = \begin{cases} 0 \text{ if } z \in [z_0, +1] \\ 1 \text{ if } z \in [-1, z_0] \end{cases}$$

and boundary conditions

$$u(t,z) = \begin{cases} 1 \text{ if } z > +1 \\ 0 \text{ if } z < -1. \end{cases}$$

Find the mixing time T, the thickness of the mixing layer, and the density profile at each time $0 < t < T$ in the symmetric case $z_0 = 0$.