Prof. McCann MAT 1502H1S Assignment 1 Due: Wednesday Jan 17

This assignment is intended primarily (a) to help students understand the course material and (b) to help the instructor understand the level and background of the students. Do not be too concerned if you find some problems difficult; just give them your best try.

1. **Riemannian geodesics and distance.** (a) Let $g_{ij}(x)$ be a smooth map from $\mathbf{R}^n$ into symmetric positive definite $n \times n$ matrices, and fix $x_0, x_1 \in \mathbf{R}^n$. If $x : [0,1] \to \mathbf{R}^n$ minimizes the energy functional
$$E[x] := \frac{1}{2} \int_0^1 g_{ij}(x(t)) \dot{x}^i(t) \dot{x}^j(t) dt$$
among all smooth curves starting at $x(0) = x_0$ and ending at $x(1) = x_1$, use calculus to show that $x(t)$ satisfies the ordinary differential equation
$$\ddot{x}^i(t) + \Gamma^i_{jk}(x(t)) \dot{x}^j(t) \dot{x}^k(t) = 0,$$
where
$$\Gamma^i_{jk} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^m}\right)$$
are the so-called Christoffel symbols. (Repeated indices are summed from 1 to n, and g^{im} denotes the matrix inverse of g_{mj}.)

b) In that case, show (using Jensen's inequality, for example) that $x(t)$ also minimizes the length functional
$$L[x] := \int_0^1 (g_{ij}(x(t)) \dot{x}^i(t) \dot{x}^j(t))^{1/2} dt$$
in the same class of curves. Its value at the miminum defines $d(x_0, x_1) := L[x]$.

c) Using (a), show by induction that if $x : [0,1] \longrightarrow \mathbf{R}^n$ minimizes $E[x]$ in the larger class of continuous, piecewise smooth curves, then $x \in C^k$ for all $k \in \mathbf{N}$.

d) Show $d(x_0, x_1) \leq d(x_0, y) + d(y, x_1)$ for all $y \in \mathbf{R}^n$ and $d(x(s), x(t)) = |s - t| d(x_0, x_1)$ for all $s, t \in [0,1]$.

2. **Functional analysis.**

Give a statement of the (a) Riesz-Markov and (b) Banach-Alaoglu theorems.

(c) Given a pair of Borel probability measures $\mu^\pm$ on a compact separable metric space (X, d), use them to show the set of non-negative joint measures $\Gamma(\mu^+, \mu^-)$ on $X \times X$ having μ^+ and μ^- for marginals is weak-$*$ compact.

(d) Let $c \in C(X)$ be a continuous function on the same space X. Show the cost functional
$$cost(\gamma) := \int_{X^2} c(x, y) d\gamma(x, y)$$
attains its minimum on $\Gamma(\mu^+, \mu^-)$.

3. **Doubly stochastic matrices.** (a) Give a statement of the Krein-Milman theorem.

(b) Use it to show that at least one of the minimizers in problem #2 above is an extreme point, meaning it fails be the midpoint of any segment in $\Gamma = \Gamma(\mu^+, \mu^-)$.

(c) Let $X = \{1, 2, \ldots, n\}$ and $\mu^\pm = \frac{1}{n} \sum_{i=1}^n \delta_i$. Show the set $\Gamma(\mu^+, \mu^-)$ corresponds to the set of non-negative $n \times n$ matrices whose columns and rows each sum to 1.

(d) In case (c) show that Γ has precisely $n!$ extreme points.