MAT 1502: Assignment #2

Due: Wednesday, Feb. 14 Note: Classes and office hours cancelled Feb 7 and 9

1. **Change of Variables:** Let M and N be sets equipped with σ-algebras $\mathcal{M}$ and $\mathcal{N}$ (of subsets). Let μ be a non-negative measure on $(M, \mathcal{M})$. Define its push-forward $t_\# \mu$ through any measurable map $t : M \longrightarrow N$ by $t_\#\mu[Y] = \mu[t^{-1}(Y)]$ for $Y \subset N$. Verify that $t_\#\mu$ is a measure on $(N, \mathcal{N})$. Then prove the change of variables formula:
$$\int_N f d(t_\#\mu) = \int_M f(t(x)) d\mu(x)$$
whenever $f : N \longrightarrow [0, \infty]$ is a measurable function on N.
Hint: Prove it first for simple functions $f = \sum_{i=1}^n \alpha_i \chi_{M_i}$.

2. **Translates and Dilates:** Fix $c(x, y) := |x - y|^2$ and a Borel probability measure μ on $\mathbf{R}^n$. Let $s(x) := \lambda x + y$, where $\lambda > 0$ and $y \in \mathbf{R}^n$, and define $\nu := s_\#\mu$ to be the translated dilate of μ. Use Brenier's theorem to prove that $s(x)$ is the optimal map between μ and ν.

3. **The Monge Problem for Two Ellipsoids:** Fix $c(x, y) := |x-y|^2$, and let μ and ν be the probability measures with distribute their mass uniformly inside two ellipsoids $E_0 \subset \mathbf{R}^n$ and $E_1 \subset \mathbf{R}^n$ of unit volume. Find the optimal map $t : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ which pushes μ forward to ν.
Hint: The solution $t(x) = Px$ is a linear map on $\mathbf{R}^n$ (affine if the ellipsoids fail to be centered at the origin). The matrix P can be expressed in terms of moments of inertia
$$\Sigma_{ij}(\mu) := \int_{\mathbf{R}^n} x_i x_j d\mu(x)$$
of μ and ν by using matrix square roots. The problem is simpler when one of the ellipsoids is a ball. The general problem requires the use of matrix square roots to solve a qaudratic matrix equation for P in terms of $\Sigma(\mu)$ and $\Sigma(\nu)$.

4. **Legendre-Fenchel transforms** (a) Compute the Legendre-Fenchel transforms of the following convex functions on $\mathbf{R}^n$: $u(x) = \langle x, Px\rangle/2$ where P is a symmetric and positive definite matrix, and b) $v(y) = |y|^p/p$ for each $p \geq 1$.
c) Set $u(x) = 0$ on a compact convex set $\Omega \subset \mathbf{R}^n$ and $u(x) = +\infty$ on $\mathbf{R}^n \setminus \Omega$. Show that
$$\max_{x \in \Omega} x \cdot y$$
is uniquely attained if and only if u^* is differentiable at y.

5. **Gluing lemma:** When X, Y, Z are compact metric spaces, use the Hahn-Banach theorem to give an alternate proof of the gluing lemma: If $\gamma^+ \in \mathcal{P}(X \times Y)$ and $\gamma^- \in \mathcal{P}(Y \times Z)$ have the same Y-marginal $(\pi^Y_\#\gamma^+ = \pi^Y_\#\gamma^-)$, there exists $\gamma \in \mathcal{P}(X \times Y \times Z)$ with γ^+ and γ^- as its marginals: $\gamma^\pm = \pi^\pm_\#\gamma$, where $\pi^+(x, y, z) = (x, y)$, $\pi^-(x, y, z) = (y, z)$ and $\pi^Y(x, y, z) = y$.