Chaos in symmetric Hamiltonians applied to some exact solutions of the semi-geostrophic approximation of 2D Incompressible Euler equations

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April 06, 2006

Abstract

Certain symmetry properties of Hamiltonian systems possessing hyperbolic fixed points with homoclinic and heteroclinic saddle connections are exploited to conclude chaotic dynamics are present under time periodic perturbations. Specifically, the theorems are applied to a set of exact solutions to the semi-geostrophic equations in an elliptical elliptical tank.

Introduction

We start this paper off by giving a brief introduction to chaos so that it is clear in the subsequent section what is meant by a *chaotic system*. We do this by finding a reference topological space that we define to be chaotic, and defining any other system to be chaotic if it is topologically equivalent to this space. We then proceed in section 2 to conclude that Hamiltonian systems that possess certain symmetries allow us to conclude chaotic dynamics are present when there are homoclinic and heteroclinic saddle connections to hyperbolic fixed points. We then give a brief introduction to the semi-geostrophic approximation that is used in oceanography and meteorology to make weather predictions. Finally, we apply the theorems developed in section 2 to a set of exact solutions to the semi-geostrohpic approximation of the 2D incompressible euler equations to conclude chaotic dynamics are present.

1 Chaos and Nonlinear Dynamics

1.1 Topological basis for chaos

Consider the following topological space. Let Σ_2 denote the set of all bi-infinite sequences $s = (\ldots, s_{-2}, s_{-1}, s_0, s_1, s_2, \ldots)$, $s_i \in \{1, 2\}$.

To define a topology we need the notion of open sets. We do this by defining a metric on Σ_2, so that (Σ_2, d) is a metric space. We define d as,

$$d(s,t) = \sum_{i\in\mathbb{Z}} \frac{|s_i - t_i|}{2^i}$$

Then we can define open neighborhoods of points $t \in \Sigma_2$ as $U_i = \{s \in X | s_k = t_k, |k| < i\}$.
Now consider the shift map $\sigma : \Sigma_2 \to \Sigma_2$, $\sigma(\ldots, s_{-2}, s_{-1}, s_0, s_1, s_2, \ldots) = \sigma(\ldots, s_{-1}, s_0, s_1, s_2, s_3, \ldots)$ The metric space (Σ_2, σ) is called the *full shift on two symbols*. So the shift map moves all elements one place to the left. Note that based on the metric above, even if two points, s and t are made arbitrarily close together (ie. $s_k = t_k$ for all $|k| < N$, for arbitrarily large N), after a finite number of iterations of the shift map, nothing can be said about how close the points will be. This is one of the main ideas that chaos is based on, which is *sensitive dependence on initial conditions.*

Definition 1 *A metric space (X, d) is said to have sensitive dependence on initial conditions if there is a $\delta > 0$ such that for any $x \in X$ and any $\epsilon > 0$, there is a $y \in X$ and some $n \in \mathbb{N}$ such that $d(x, y) < \epsilon$ and $d(f^n(x), f^n(y)) > \delta$*

Now should we say that a system is chaotic when it exhibits sensitive dependence to initial conditions? That this definition for chaos would be insufficient is shown in the following common example. Let $M = (0, \infty)$ and define $f(x) = (1+\mu x)$, $\mu > 0$. Then (M, d), where d is the standard euclidian metric on $\mathbb{R}$ exhibits sensitive dependence on initial conditions, but each point converges to ∞. (Example taken from [5]).

We want our definition of chaos incorporate the difficulty in making any sort of long term predictions about the behavior. Since we have already said that points close together must move away from each other quickly, we now say that any two points will at some time be arbitrarily close together. In this way we avoid the possibility of all points moving away from each other and approaching the same limit. Hence we want our dynamical system to have a *dense orbit*, meaning that there is some $x \in X$ such that for all $y \in X$, given any $\epsilon > 0$ there exists an $N > 0$ such that $d(f^N(x), y) < \epsilon$. Note that that (σ, Σ_2) possesses a dense orbit (see [5]). So we want a way of saying that a system is chaotic if it is in some sense topologically equivalent to the metric space (Σ_2, σ). To make this precise, we define the notion of a topological conjugate,

Definition 2 *Two maps $f : A \to A$ and $g : B \to B$ are topologically conjugate if there exists a homeomorphism $\phi : A \to B$ such that $\phi \circ f = g \circ \phi$*

Now we can make precise what we mean by a chaotic system, in the context of a planar diffeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$, which is what will be the only map for this paper.

Definition 3 *A planar diffeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ with the standard euclidian metric in $\mathbb{R}^2$ is said to be chaotic if it is topologically conjugate to the full shift on 2 symbols, (Σ_2, σ).*

Hence the full shift on two symbols is our topological basis for chaos, and determining if a system is chaotic will amount to establishing the existence of a topological conjugate to the full shift on two symbols, (Σ_2, σ).

1.2 Chaos in dynamical systems

Although the disucssion above was limited to iterative maps, we can still apply it in the case of continuous flows, given by a first order ODE such as

$$\dot{x} = f(x) \quad (1.21)$$

To resolve this we extend our phase space by one dimension, and consider time as a coordinate, so that our system becomes

$$\dot{x} = f(x,t) = f(x) \quad (1.22)$$

$$\dot{t} = 1 \quad (1.23)$$

Since f is autonomous, we can arbitrarily choose a period of $T = 1$ and wrap our phase space onto $\mathbb{R}^2 \times S^1$. By considering the *Poincare map*, $P_{t_0} : \Sigma_{t_0} \to \Sigma_{t_0}$ where $\Sigma_{t_0} = \{(x, t_0) \in \mathbb{R}^2 \times S^1\}$, ie. the projection of points $(x,t) \in \mathbb{R}^2 \times S^1$ onto the plane $t = t_0$, we obtain an iterative map of the flow (1.22)-(1.23). Our goal from now on will be to establish when the Poincare map is a topological conjugate to the full the shift on two symbols, and we will say that a system such as (1.21) is chaotic if it possesses a Poincare map that is such a topological conjugate.

Smale [6] came up with a precise sufficient condition for a planar diffeomorphism to be topologically conjugate to the shift on two symbols. We state it below.

Theorem 1 *(Smale Birkhoff) Suppose f is a diffeomorphism with a hyperbolic fixed point p and a corresponding transversal homoclinic point q. Then some iterate f^n has a hyperbolic invariant set I on which it is topologically equivalent (conjugate) to the bi-infinite shift on two symbols.*

By a *transversal* intersection of the stable and unstable manifolds of the hyperbolic fixed point p, denoted $W_0^u(p)$ and $W_0^s(p)$ respectively is meant that the two manifolds intersect such a way that the union of the tangent spaces, $T_u(p)$ and $T_s(p)$ for the unstable and stable manifolds respectively, spans $\mathbb{R}^2$ at the point of intersection $q \in W_0^u(p) \cap W_0^s(p)$.

Of course in the case that $f(x)$ is autonomous, the Poincare map is of little assistance since the projection onto any plane normal to the time axis is the same for all times. However, consider what happens when we introduce a time periodic perturbation of (1.21), $\epsilon g(x,t)$ where $g(x, t+T) = g(x,t)$ for fixed

Figure 1: A homoclinic tangle. There are an infinite number of transversal intersection of the manifolds.

period T, $0 < T < \infty$. Given a hyperbolic fixed point $p \in \mathbb{R}^2$ of f, if ϵ is sufficiently small, the hyperbolic fixed point of the Poincare map, p, perturbs to a nearby hyperbolic fixed point $p_\epsilon = p_0 + O(\epsilon)$ (precise statement and proof are given in [1], section 4.5).

Now treating our system as

$$\dot{x} = f(x) + \epsilon g(x,t) \quad (1.24)$$

$$\dot{t} = 1 \quad (1.25)$$

we turn our system into a non-autonomous flow in $\mathbb{R}^2 \times S^1$. So the idea of a poincare map is more useful in this scenario. Since we can imagine setting up planes, perpendicular to the time axis at every $t_0 + nT$ for $n \in \mathbb{N}$. The plot of the poincare map will be the intersection points of the solution curves $(x(t), t)$ in $\mathbb{R}^2 \times \mathbb{R}$ with the planes Σ_{t_0}.

We provide an intuitive motivation for the Smale-Birkhoff theorem. For precise details the reader is referred to [1]. From before, the hyperbolic fixed point p of the unperturbed poincare map gets perturbed to a nearby hyperbolic fixed point p_ϵ. Denote $W^u_\epsilon(p_\epsilon)$ and $W^s_\epsilon(p_\epsilon)$ as the unstable and stable manifolds of the perturbed system (1.24)-(1.25) respectively at the new hyperbolic fixed point p_ϵ. Now what happens if these manifolds intersect transversally? Consider the poincare map acting on the point of intersection $q \in \mathbb{R}^2$. Then by the invariance of the manifolds $P^n_{t_0}(q) \in W^u_\epsilon(p_\epsilon) \cap W^s_\epsilon(p_\epsilon)$ for all $n \in \mathbb{Z}$. Hence the iterative map produces an infinite number of intersection points, as shown in Figure 1, and the manifolds *tangle* around each other. This is where the term *homoclinic tangle* comes from.

By associating a 1 or a 0 depending on the path chosen, as shown in Figure 1, it is clear why there is a bijection between points in the map and points in the space of all by infinite sequences Σ. Also, a forward iteration of the poincare map is equivalent to the shift operator acting on the space of bi-infinite sequences. A rigorous justification of these arguments can be found in [1].

Now we have seen why transversal intersections are important, and have defined what we mean by *chaotic* while also finding a sufficient condition for chaos to occur (transversal intersection of stable and unstable manifolds of hyperbolic fixed points).

The problem is, how can one go about showing that something as complicated as transversal intersections occur? This is where melnikov comes to the rescue. Melnikov [6] stated the following sufficient condition for time periodic perturbed system with a saddle connection.

We define the Melnikov function

$$M(t_0) = \int_{-\infty}^{\infty} f(q_0(t)) \wedge g(q_0(t), t + t_0)dt$$

Figure 2: How a symmetrical perturbation will imply a transversal intersection of the stable and unstable manifolds

Where $q_0(t)$ denotes the paramaterized saddle connection between two hyperbolic fixed points of the unperturbed system. We state Melnikov's Theorem

Melnikov's Theorem *Given that f,g are C^r, $r \geq 2$, and for $\epsilon = 0$, (1.1) possesses a homoclinic saddle connection to a hyperbolic fixed point p. Then for ϵ sufficiently small if $M(t_0)$ has simple zeros then $W^s_\epsilon(p_\epsilon)$ and $W^u_\epsilon(p_\epsilon)$ intersect transversally.*

The proof of the above theorem and derivation of the Melnikov integral is ubiquitous and can be found in [1].

The idea is that the integral measures the separation of the perturbed stable and unstable manifolds. If a simple zero is found, by the above theorem there is a transversal intersection. Hence by the Smale-Birkoff theorem (Theorem 1), the Poincare map possesses an invariant set on which some iteration of the map is topologically conjugate to the shift on two symbols, and hence chaotic as defined in the preceding section.

We now wish to give an intuitive idea of how exploiting symmetry of Hamiltonian systems can allow one to conclude systems are chaotic by Melnikov's Theorem.

1.3 Hamiltonian Symmetry leading to chaos

Assume for the moment that we have a sufficiently smooth Hamiltonian, $H : \mathbb{R}^2 \to \mathbb{R}$, such that $H(-x, y) = H(x, y)$, so that our Hamiltonian is symmetric about the y-axis. This leads to

$$f_2(-x, y) = \frac{\partial H}{\partial x}(-x, y) = -\frac{\partial H}{\partial x}(x, y) = -f_2(x, y))$$

$$-f_1(-x, y) = \frac{\partial H}{\partial y}(-x, y) = \frac{\partial H}{\partial y}(x, y) = -f_1(x, y)$$

where $f(x, y) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix}$ And imagine we had a homoclinic orbit as shown in Figure 3. Then consider a perturbation $\epsilon g(x, y, t) = \epsilon \begin{pmatrix} g_1(x, y, t) \\ g_2(x, y, t) \end{pmatrix}$ as was used above but such that $g_1(x, y, t) = g_1(-x, y, t)$ and $g_2(x, y, t) = -g_2(-x, y, t)$ so that the perturbation shares the same symmetry as the system. Then for a particular value of t_0, we look at the perturbed poincare map $P^\epsilon_{t_0} : \Sigma_{t_0} \to \Sigma_{t_0}$. The stable and unstable manifolds will break apart. If neither the stable or unstable manifold intersected the y-axis again, then they would clearly not remain within $O(\epsilon)$ of each other, which would contradict the fact that the perturbed stable and unstable manifolds stay within $O(\epsilon)$ of the unperturbed manifolds, a result proven in [1], p. 186. Hence because of the symmetry of the Hamiltonian and the perturbation, the Poincare slice, Σ_{t_0} must look something like Figure 2, and so we get a transversal intersection of the manifolds in this case on the y-axis. This suggests that by looking at symmetry alone one can conclude that certain systems are chaotic. This is made precise in the next section.

2 Main results

2.1 Chaos in the case of a symmetrical perturbation of a symmetrical Hamiltonian system possessing a homoclinic saddle connection to a hyperbolic fixed point

In Theorem 2, we present sufficient symmetry conditions on the Hamiltonian, and perturbation of the Hamiltonian in the case that the system possesses a homoclinic orbit. The theorem allows us to guarantee our system is chaotic in the sense of section 1, without the need to evaluate the melnikov integral explicitly, given in Melnikov's theorem in section 1.

Theorem 2 *: (Homoclinic case) Given the planar, Hamiltonian system*

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^2 \qquad (2.0)$$

with Hamiltonian H, where (2.0) *has a homoclinic point p lying on the y-axis. If*

(a) $q_0 : \mathbb{R} \to \mathbb{R}^2$, the homoclinic orbit, crosses the y-axis at some point other than p

(b) $f \in C^r(D), r \geq 2$ for some bounded domain D such that $q_0(t) \in D$ for all $t \in \mathbb{R}$ and $f(q_0(t)) \in L^2(\mathbb{R})$ (hereafter referred to as 'sufficiently smooth and integrable')

(c) $H(-x, y) = H(x, y)$ (we say the Hamiltonian has reflection symmetry about y)

★ *Then for any function $h(x, y)$ that also satisfies* (*b*) *such that*

$$h_1(-x, y) = h_1(x, y)$$

$$h_2(-x, y) = -h_2(x, y)$$

the melnikov integral,

$$M(t_0) = \int_{-\infty}^{\infty} f(q_0(t)) \wedge g(q_0(t), t + t_0)dt$$

Figure 3: Homoclinic orbit symmetrical about the y-axis (Case 1)

Figure 4: Homoclinic orbit symmetrical about the y-axis (Case 2)

where

$$g(x, y, t) = \epsilon \cos(kt) h(x, y)$$

has simple zeroes for some $k \in \mathbb{R}$ *and* ϵ *sufficiently small. It follows that the system* (2.0) *is chaotic under small perturbations.*

Proof: First note that

$$f_2(-x, y) = \frac{\partial H}{\partial x}(-x, y) = -\frac{\partial H}{\partial x}(x, y) = -f_2(x, y) \quad (2.1a)$$

$$-f_1(-x, y) = \frac{\partial H}{\partial y}(-x, y) = \frac{\partial H}{\partial y}(x, y) = -f_1(x, y) \quad (2.1b)$$

Since the homoclinic trajectory cross the y-axis at some point other than p, based on the symmetry of f in $(2.1a) - (2.1b)$, we must have an orbit as shown in Figure 3 or Figure 4, depending on whether the homoclinic orbit crosses the y-axis above or below the homoclinic point p. It is of no loss of generality to assume the orbit takes the form in Figure 3, as will be obvious in the proof.

Referring to this figure, paramaterize the homoclinic orbit as $q_0(t - t_0)$ so that at time $t = t_0$ the orbit cross the y-axis at the point $q_0(0)$.

We now compute the wedge product that is in the melnikov integral.

$$\begin{pmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{pmatrix} \wedge \begin{pmatrix} g_1(\mathbf{x}, t) \\ g_2(\mathbf{x}, t) \end{pmatrix} = \epsilon \cos(kt)[f_1(x, y)h_2(x, y) - f_2(x, y)h_1(x, y)] \quad (2.12)$$

Now based on the symmetry arguments above, it is clear that (2.12) has odd symmetry about the y-axis. Then we have

$$f(q_0(t)) \wedge g(q_0(t), t+t_0) = \epsilon \cos(k(t+t_0))[f_1(q_0(t))h_2(q_0(t)) - f_2(q_0(t))h_1(q_0(t))] \quad (2.13)$$

Where we can ignore the ϵ term in the melnikov integral since we are trying to find it's zeros. Now from (2.12) and the fact that we paramaterized $q_0(t)$ so that $q_0(0)$ lies on the y axis, it is clear that (3) considered as a function of $q_0(t)$),

$$[f_1(q_0(t))h_2(q_0(t)) - f_2(q_0(t))h_1(q_0(t))]$$

is an odd function of time. Now from (2.13), upon expanding the cosine sum, one notes that the integral is broken up into two parts, and the part

$$\cos(kt_0) \int_{-\infty}^{\infty} \cos(kt)[f_1(q_0(t))h_2(q_0(t)) - f_2(q_0(t))h_1(q_0(t))]dt$$

vanishes since it is an integral from $-\infty$ to $+\infty$ of an odd function. Hence the part that remains is

$$M(t_0) = \sin(kt_0)\int_{-\infty}^{\infty} \sin(kt)[f_1(q_0(t))h_2(q_0(t)) - f_2(q_0(t))h_1(q_0(t))]dt := \sin(kt_0)M(k) \quad (2.14)$$

But (2.14) is just the sin fourier transform of an $L^1(\mathbb{R})$ function. This follows since $f(q_0(t)), h(q_0(t)) \in L^2(\mathbb{R})$ and so $f(q_0(t))h(q_0(t)) \in L^1(\mathbb{R})$.

Hence based on the properties of the sin fourier transform of an $L^1(\mathbb{R})$ function [2, pp. 120-131] there exists an interval $[k_1, k_2]$ where $M(k)$ is nonzero. For all $k \in [k_1, k_2]$, $M(t_0)$ has simple zeros since $\sin(kt_0)$ does. Hence by Melnikov's theorem, given in Section 1, the system (2.0) is chaotic. □

We now extend the argument to the case of two hyperbolic fixed points p_1, p_2, $p_1 \neq p_2$, such that p_1 and p_2 are connected by a heteroclinic saddle connection.

2.2 Chaos in the case of a symmetrical perturbation of a Hamiltonian system with a heteroclinic saddle connection between two hyperbolic fixed points

Theorem 3 : *(Heteroclinic case) Given the planar, Hamiltonian system*

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^2 \qquad (1)$$

with Hamiltonian H, where (1) has a heteroclinic saddle connection between two hyperbolic fixed points p_1 and p_2 both lying on the y-axis. If

(a) p_1 and p_2 are reflections across the x-axis of each other.

(b) $f \in C^r(D), r \geq 2$ for some bounded domain D such that $q_0(t) \in D$ for all $t \in \mathbb{R}$ and $f(q_0(t)) \in L^2(\mathbb{R})$ (hereafter referred to as 'sufficiently smooth and integrable')

(c) $H(-x, y) = H(x, y)$ and $H(x, -y) = H(x, y)$ (symmetry across both x and y-axes)

★ *Then for any function $h(x, y)$ that also satisfies (b) such that*

$$h_1(x, -y) = -h_1(x, y)$$

$$h_2(x, -y) = h_2(x, y)$$

the melnikov integral*,*

$$M(t_0) = \int_{-\infty}^{\infty} f(q_0(t)) \wedge g(q_0(t), t + t_0)dt$$

Figure 5: Heteroclinic orbit symmetrical about the y-axis and x-axis

where

$$g(x, y, t) = \epsilon \cos(kt) h(x, y)$$

has simple zeroes for some $k \in \mathbb{R}$ *and* ϵ *sufficiently small.*

Proof: First note that

$$f_2(x, -y) = \frac{\partial H}{\partial x}(x, -y) = \frac{\partial H}{\partial x}(x, y) = f_2(x, y) \quad (2.2a)$$

$$-f_1(x, -y) = \frac{\partial H}{\partial y}(x, -y) = -\frac{\partial H}{\partial y}(x, y) = f_1(x, y) \quad (2.2b)$$

Since the heteroclinic trajectory crosses the x-axis at some point, based on the symmetry of f in $(2.2a) - (2.2b)$, we must have an orbit as shown in Figure 5. Note that we are guaranteed to have a second heteroclinic saddle connection between p_1 and p_2 on the other side of the y-axis by the symmetry of the hamiltonian in (c).

Referring to this figure, parameterize the homoclinic orbit as $q_0(t - t_0)$ so that at time $t = t_0$ the orbit crosses the x-axis at the point $q_0(0)$, shown in Figure 5.

We now compute the wedge product that is in the melnikov integral.

$$\begin{pmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{pmatrix} \wedge \begin{pmatrix} g_1(\mathbf{x}, t) \\ g_2(\mathbf{x}, t) \end{pmatrix} = \epsilon \cos(kt)[f_1(x, y)h_2(x, y) - f_2(x, y)h_1(x, y)] \quad (2.22)$$

Now based on the symmetry arguments above, it is clear that (2.22) has odd symmetry about the x-axis. Then we have

$$f(q_0(t)) \wedge g(q_0(t), t + t_0) = \epsilon \cos(k(t + t_0))[f_1(q_0(t))h_2(q_0(t)) - f_2(q_0(t))h_1(q_0(t))] \quad (2.23)$$

Where we can ignore the ϵ term in the melnikov integral since we are trying to find it's zeros. Now from (2.22) and the fact that we paramaterized $q_0(t)$ so that $q_0(0)$ lies on the x axis, it is clear that (2.22) considered as a function of $q_0(t)$),

$$[f_1(q_0(t))h_2(q_0(t)) - f_2(q_0(t))h_1(q_0(t))]$$

is an odd function of time. Now from (2.23), upon expanding the cosine sum, one notes that the integral is broken up into two parts, and the part

$$\cos(kt_0) \int_{-\infty}^{\infty} \cos(kt)[f_1(q_0(t))h_2(q_0(t)) - f_2(q_0(t))h_1(q_0(t))]dt$$

vanishes since it is an integral from $-\infty$ to $+\infty$ of an odd function. Hence the part that remains is

$$M(t_0) = \sin(kt_0)\int_{-\infty}^{\infty} \sin(kt)[f_1(q_0(t))h_2(q_0(t)) - f_2(q_0(t))h_1(q_0(t))]dt := \sin(kt_0)M(k) \quad (2.24)$$

But (2.24) is just the sin fourier transform of an $L^1(\mathbb{R})$ function. This follows since $f(q_0(t)), h(q_0(t)) \in L^2(\mathbb{R})$ and so $f(q_0(t))h(q_0(t)) \in L^1(\mathbb{R})$.

Hence based on the properties of the sin fourier transform of an $L^1(\mathbb{R})$ function [2, pp. 120-131] there exists an interval $[k_1, k_2]$ where $M(k)$ is nonzero. For all $k \in [k_1, k_2]$, $M(t_0)$ has simple zeros since $\sin(kt_0)$ does. By symmetry, the perturbed heteroclinic saddle connection on the other side of the y-axis also has simple zeros and hence transversal intersections of the stable and unstable manifolds. It follows from the *Heteroclinic theorem* given in [3], which is an extension of the Smale-Birkhoff theorem given in section 1 to heteroclinic saddle connections to hyperbolic fixed points, that the system is chaotic. □

2.3 Symmetry conditions in the case of polar coordinates

We note that the symmetry condition in Theorem 1 when expressed in polar coordinates,

$$\begin{pmatrix} \dot{r} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} f_1(r,\phi) \\ f_2(r,\phi) \end{pmatrix}$$

the required symmetry condition is

$$f_1(r, \pi - \phi) = -f_1(r, \phi) \quad (2.31)$$

$$f_2(r, \pi - \phi) = f_2(r, \phi) \quad (2.32)$$

For Theorem 2, we need the above symmetry condition along with,

$$f_1(r, -\phi) = -f_1(r, \phi) \quad (2.33)$$

$$f_2(r, -\phi) = f_2(r, \phi) \quad (2.34)$$

2.4 Conclusion

Hence we have reduced the problem of determining that a periodically perturbed Hamiltonian system is chaotic to checking for symmetries in the Hamiltonian. Avoiding the need to evaluate the melnikov integral explicitly makes checking for chaos much simpler in many instances. In section 4 we apply these theorems to the Semi-Geostrophic equations, which are introduced in the following section.

3 Semi-Geostrophic equations

For an incompressible fluid belonging to some domain $D \subset \mathbb{R}^3$ that is uniform along one of its axes, the Navier Stokes equations can be restricted to a domain $Y \subset \mathbb{R}^2$. They become,

$$(\partial_t + \mathbf{v} \cdot \nabla)\mathbf{v} + 2\Omega J\mathbf{v} = -\frac{1}{\rho}\nabla P \quad (3.1)$$

$$\nabla \cdot \mathbf{v} = 0 \quad (3.2)$$

$$\mathbf{v} \cdot \hat{\mathbf{n}} = 0 \quad (3.3)$$

Where $\hat{\mathbf{n}}$ is the normal to the boundary Y, and $\mathbf{v}$ is a function of space and time, so $\mathbf{v} : Y \times [0, \infty) \to \mathbb{R}^2$. These equations are the *2D Incompressible Euler Equations* where

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

These equations can be re-written in terms of a stream function since because of the incompressibility condition, we can say

$$\mathbf{v}(\mathbf{x}, t) = J\nabla\psi(\mathbf{x}, t) \quad (3.4)$$

The euler equations become,

$$\nabla\frac{\partial\psi}{\partial t} + (D^2\psi + I)J\nabla\psi - J\nabla P = 0 \quad (3.5)$$

In large scale atmospheric flow, the acceleration terms in the Euler equations can be neglected entirely which gives Q=P. The *semi-geostrohpic* approximation to the euler equations involves approximating the small terms in (3.5) only by $Q \backsim P$ so that we get

$$\nabla\frac{\partial P}{\partial t} + (D^2 P + I)J\nabla\psi - J\nabla P = 0 \quad (3.6)$$

Now McCann and Oberman [4] considered the a fluid restricted to an elliptical tank, so that the domain Y in the above euler equations is a fixed elliptical boundary, but the evolution is considered under the SG approximation given by (3.6). They came up with a set of exact solutions that model the evolution of the ellipse in dual coordinates, $a(t)$ and $\theta(t)$ which represent the aspect ratio and inclination with respect to the coordinate axes of the dual ellipse, given below:

$$\frac{da}{dt} = -\frac{2\lambda\sinh(\phi)}{z(a, \theta; \phi)} a\sin(2\theta) \quad (3.7)$$

$$\frac{d\theta}{dt} = 1 - \frac{\lambda}{z(a, \theta; \phi)}(\cosh(\phi) + \frac{a^2+1}{a^2-1}\sinh(\phi)\cos(2\theta)) \quad if \quad a \neq 1 \quad (3.8)$$

$$\frac{d\theta}{dt} = \frac{1}{2}(1 - \frac{\lambda\cosh(\phi)}{2\cosh(\phi/2)}) \quad if \quad a = 1 \quad (3.9)$$

Figure 6: Phase diagram for (3.5) on axes $(x, y) = (\log(a)\cos(\theta), \log(a)\sin(\theta)$ with above values of λ and ϕ. The homoclinic orbit is shown in bold (taken courtesy of [4])

$$z(a, \theta; \phi) = \sqrt{2 + (a + \frac{1}{a})\cosh(\phi) + (a - \frac{1}{a})\sinh(\phi)\cos(2\theta)} \quad (3.10)$$

$$H(r, \theta) = \lambda^2 s + r - \lambda(2 + 2rs + 2\cos(2\theta)\sqrt{(r^2-1)(s^2-1)})^{1/2} \quad (3.11)$$

Where λ is a parameter measuring the area of the elliptical tank and ϕ represents the aspect ratio of the elliptical tank.

The semi-geostrophic approximation gives more accurate meterological predictions and is considered less turbulent than the 2D euler equations. It is consequently not obvious that chaotic dynamics should be present in solutions to these equations. In the following section we consider how a perturbation of the area of the ellipse leads to chaotic dynamics, and we apply the theorems developed in the previous section to do so.

4 Chaos in the Semi-Geostrophic equations

We now apply the results of section 2, to the solutions to the semi-geostrophic equations (3.7)-(3.11) and determine that chaotic dynamics are present.

Proposition 1 *The system (3.7)-(3.11) above is chaotic for* $(\lambda, \phi) = (3, 0.2068522964)$.

Proof: The proof is broken down into four steps.
Step 1: Determine that the hamiltonian satisfies the symmetry constraints given in Theorem 2

This is easy since,

$$\begin{aligned} H(r, \pi - \theta) &= \lambda^2 s + r - \lambda(2 + 2rs + 2\cos(2\pi - 2\theta)\sqrt{(r^2-1)(s^2-1)})^{1/2} \\ &= \lambda^2 s + r - \lambda(2 + 2rs + 2\cos(2\theta)\sqrt{(r^2-1)(s^2-1)})^{1/2} \end{aligned}$$

since $\cos(2\pi - 2\theta) = \cos(2\theta)$ This symmetry is clear when one looks at Figure 6. Hence condition (c) of Theorem 2 is satisfied.
Step 2: Determine that a homoclinic point exists on the y-axis and that this orbit intersects the y-axis at some other point.

That a unique hyperbolic fixed point exists on the positive half of the y-axis is

Figure 7: Homoclinic orbit magnified from Figure 6, and $a(\infty)$, $a(0)$ shown. Plotted on axes $(x, y) = (a\cos(\theta), a\sin(\theta))$

shown in [4], (Theorem 1.4). As can be seen from Figure 6 (taken from [4]), for these values of λ and ϕ, the hyperbolic fixed point possesses a homoclinic orbit that crosses the y-axis at some other point, $\log(a(0)) > 0$ (shown in Figure 6). Hence condition (a) of Theorem 2 is satisfied

Step 3: Determine that $\frac{da}{dt}(q_0(t))$ and $\frac{d\theta}{dt}(q_0(t))$ are sufficiently smooth and integrable on $\mathbb{R}$, ie. restricted to the homoclinic orbit, $q_0(t)$.

Let $\sinh(0.2068522964) = 1.2015 := \beta$ and $\cosh(0.2068522964) = 0.2083 := \gamma$. We show that $\frac{da}{dt} \in L^2(\mathbb{R})$ when restricted to the homoclinic orbit. In Figure 7, the points $a(0)$ and $a(\infty)$ are shown. It is clear from Figure 6 that $\log(a(\infty)) > 0$ and $\log(a(0)) > 0$, and hence $a(\infty) > 1$ and $a(0) > 1$.

$$\int_{-\infty}^{\infty} |\frac{da}{dt}^2|da = 2\int_{a(0)}^{a(\infty)} \frac{4\lambda^2\beta^2}{z^2} a^2 \sin(2\theta)^2 da \quad (4.1)$$

Where $\theta = \theta(a)$ is an implicit function of a that parameterizes the homoclinic orbit. Now

$$z \geq \sqrt{2 + (a + \frac{1}{a})\gamma - (a - \frac{1}{a})\beta}$$

$$\Rightarrow \frac{1}{z^2} \leq \frac{1}{2 + (a + \frac{1}{a})\gamma - (a - \frac{1}{a})\beta} \quad (4.2)$$

$$\int_{a(0)}^{a(\infty)} \frac{4\lambda^2\beta^2}{z^2} a^2 \sin(2\theta)^2 da \leq \int_{a(0)}^{a(\infty)} \frac{4\lambda^2\beta^2}{2 + (a + \frac{1}{a})\gamma - (a - \frac{1}{a})\beta} a^2 da$$

The only values of a where the last integrand is discontinuous occur for $\alpha \in \mathbb{C}$ (which can be seen by solving it numerically) and $\alpha = 0$, which is not included in our domain . Hence the last integral is the integral of a continuous function on a compact domain, $[a(0), a(\infty)]$. Consequently

$$\int_{a(0)}^{a(\infty)} \frac{4\lambda^2\beta^2}{2 + (a + \frac{1}{a})\gamma - (a - \frac{1}{a})\beta} a^2 da < \infty$$

Hence $\frac{da}{dt}(q_0(t)) \in L^2(\mathbb{R})$.

Now we show the same thing for $\frac{d\theta}{dt}$.

$$\int_{a(0)}^{a(\infty)} |\frac{d\theta}{dt}^2|da = \int_{a(0)}^{a\infty} (1 - \frac{\lambda}{z}(\gamma + \frac{a^2+1}{a^2-1}\beta\cos(2\theta)))^2 da$$

From equation (4.2) and the fact that $\cos(2\theta) \geq -1$,

$$\int_{a(0)}^{a\infty} (1 - \frac{\lambda}{z}(\gamma + \frac{a^2+1}{a^2-1}\beta\cos(2\theta)))^2 da \leq \int_{a(0)}^{a(\infty)} (1 + \frac{1}{z}(\gamma + \frac{a^2+1}{a^2-1}))^2 da$$

$$\leq \int_{a(0)}^{a(\infty)} (1 + \frac{\lambda}{\sqrt{2 + (a + \frac{1}{a}\gamma + \beta(a - \frac{1}{a}))}}(\gamma + \frac{a^2+1}{a^2-1}))^2 da$$

And since $a = 0$ is not on the homoclinic orbit and

$$2 + (a + \frac{1}{a})(1.0215) + (0.2083)(a - \frac{1}{a}) = 0$$

has only complex solutions, the only possible singularity could exist at $a = 1$, which is not included in the domain $[a(0), a(\infty)]$. Hence, since the function is being integrated over a compact domain, we have that

$$\int_{a(0)}^{a(\infty)} (1 + \frac{\lambda}{\sqrt{2 + (a + \frac{1}{a}\gamma + \beta(a - \frac{1}{a}))}}(\gamma + \frac{a^2+1}{a^2-1}))^2 da < \infty$$

It follows that $\frac{d\theta}{dt}(q_0(t)) \in L^2(\mathbb{R})$.

That the functions $\frac{da}{dt}$ and $\frac{d\theta}{dt}$ are C^∞ is clear except at the point $a = 1$, which is not included in homoclinic orbit. Since we only need local smoothness from condition (b) in theorem 1, we can choose our bounded domain D to be the homoclinic orbit itself. Hence the hamiltonian H is sufficiently smooth on a bounded domain D containing the homoclinic orbit, and the vector field is L^2 integrable over the homoclinic orbit. Hence we have shown that condition (b) of Theorem 2 is satisfied.

Step 4: Note that H depends on the parameter λ which corresponds to the area of the ellipse. Treating $\frac{da}{dt}$ and $\frac{d\theta}{dt}$ as functions of λ, we consider perturbing the area of the physical ellipse.

$$\frac{da}{dt}(a, \theta, \lambda + \epsilon) = \frac{da}{dt}(a, \theta, \lambda) + \frac{\partial}{\partial\lambda}(\frac{da}{dt}(a, \theta, \lambda))\epsilon$$

$$\frac{d\theta}{dt}(a, \theta, \lambda + \epsilon) = \frac{d\theta}{dt}(a, \theta, \lambda) + \frac{\partial}{\partial\lambda}(\frac{d\theta}{dt}(a, \theta, \lambda))\epsilon$$

And from $(3.1) - (3.3)$ we have that

$$\frac{\partial}{\partial\lambda}(\frac{da}{dt}) = -\frac{2\sinh(\phi)}{z(a, \theta; \phi)} a\sin(2\theta)$$

and

$$\frac{\partial}{\partial\lambda}(\frac{d\theta}{dt}) = -\frac{1}{z(a, \theta; \phi)}(\cosh(\phi) + \frac{a^2+1}{a^2-1}\sinh(\phi)\cos(2\theta))$$

Figure 8: Phase plot of (1) with above values of ϕ and λ. Heteroclinic orbit is shown in bold. Plotted on axes $(x, y) = (\log(a)\cos(\theta), \log(a)\sin(\theta))$

Notice that both of these functions have the same symmetries as $\frac{da}{dt}$ and $\frac{d\theta}{dt}$ which can be seen by plugging in $\theta' = \pi - \theta$ and comparing to equations $(3.1) - (3.5)$. The smoothness and $L^2(\mathbb{R})$ are clear from the smoothness and L^2 integrability of $\frac{da}{dt}$ and $\frac{d\theta}{dt}$. Hence they satisfy the requirements of ★ in Theorem 2.

Based on the above 4 conclusions combined with Theorem 2, the system (3.7)-(3.11) is chaotic when $(\lambda, \phi) = (3, 0.2068522964)$. □

Proposition 2 *The system (3.7)-(3.11) above is chaotic for* $(\lambda, \phi) = (3, 0.1368522964)$.

Proof:

The proof is broken down into four steps as in the homoclinic case.

Step 1: Determine that the hamiltonian satisfies the symmetry constraints given in Theorem 3.

. The first symmetry was shown in Proposition 1, the second corresponds to the symmetry in (2.33)-(2.34) and is satisfied since the Hamiltonian satisfies,

$$H(r, -\theta) = \lambda^2 s + r - \lambda(2 + 2rs + 2\cos(-2\theta)\sqrt{(r^2-1)(s^2-1)})^{1/2}$$

$$= \lambda^2 s + r - \lambda(2 + 2rs + 2\cos(2\theta)\sqrt{(r^2-1)(s^2-1)})^{1/2} = H(r, \theta)$$

since $\cos(-2\theta) = \cos(2\theta)$. This symmetry is clear when one looks at Figure 8. Hence condition (c) of Theorem 3 is satisfied.

Step 2: Determine that there are two fixed point p_1 and p_2, that are located on the y-axis and reflections of eachother across the x-axis, with a heteroclinic saddle connection connecting them crossing the x-axis.

That two hyperbolic fixed points exist on the y-axis and are reflections of eachother is shown in [4] (Theorem 1.4) and as can be seen in Figure 8. A heteroclinic saddle connection connects the two points and crosses the x-axis. Hence condition (a) of Theorem 3 is satisfied.

Step 3: Determine that $\frac{da}{dt}$ and $\frac{d\theta}{dt}$ are sufficiently smooth and integrable on $q_0(t)$, the heteroclinic saddle connection on the positive part of the x-axis.

We show that $\frac{da}{dt} \in L^2(\mathbb{R})$ when restricted to the heteroclinic orbit. In figure 5, the points $a(0)$ and $a(\infty)$ are shown as $q_0(0)$ and p_1 respectively. From Figure 8 it is clear that $\log(a(\infty)) > \log(a(0)) > 0$, and hence that $a(\infty) > a(0) > 1$.

By the arguments of Step 3, in Proposition 1, since we have shown the heteroclinic orbit in Figure 8 that is in bold contains no points of discontinuity, we have that $\frac{da}{dt}(q_0(t))$ and $\frac{d\theta}{dt}(q_0(t))$ are sufficiently smooth and integrable in the sense of Theorem 3 (b).

Step 4: The perturbation proceeds exactly in the same way as in Step 4 of Proposition 1, and the smoothness and integrability come consequently in the same way.

Based on the above 4 conclusions combined with Theorem 3, the system (3.7)-(3.11) is chaotic when $(\lambda, \phi) = (3, 0.2068522964)$. $\square$.

5 References

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