FACULTY OF ARTS AND SCIENCE

University of Toronto

FINAL EXAMINATIONS, DECEMBER 2001

MAT 327H1F - Introduction to Topology

Duration - 3 hours

Instructor: Prof. R. McCann

No aids allowed.

Questions are weighted as indicated. Total marks for this paper: 225.

1. (40 marks) Prove or give counterexamples to the following TRUE or FALSE statements:

a) Homotopy defines an equivalence relation among continuous maps $f : X \longrightarrow Y$.

b) Let (X, d) and (Y, ρ) be metric spaces. If X is compact, then any continuous map $f : X \longrightarrow Y$ is uniformly continuous.

c) Let $p : E \longrightarrow B$ be a covering map with $p(e_0) = b_0$. If E path connected, then the lifting correspondence $\phi : \pi_1(B, b_0) \longrightarrow p^{-1}(b_0)$ is one-to-one.

d) Let $f : X \longrightarrow Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

2. (60 marks) Identify the fundamental group of each of the following topological spaces and give an outline of how you would prove it.

a) the circle $\mathbf{S}^1$;
b) the 3-sphere $\mathbf{S}^3$

c) the twice punctured sphere $\mathbf{S}^2 \setminus \{n, s\}$, with $s \neq n \in \mathbf{S}^2$;
d) the 3-torus $\mathbf{T}^3 = \mathbf{R}^3/\mathbf{Z}^3$;

e) the boundary $\partial\Omega$ of a bounded convex domain $\Omega \subset \mathbf{R}^3$;
f) projective space P^3.

BONUS (5-10 marks): the figure eight space.

3. (25 marks) Let $\{S_\beta\}_{\beta \in B}$ and $\{T_\beta\}_{\beta \in B}$ be two families of topologies on X.

a) Show that $\cap_{\beta \in B} T_\beta$ is a topology on X.

b) Show there is a coarsest topology containing $\cup_{\beta \in B} S_\beta$ by using part a).

4. (25 marks) Let (X, d) be a compact metric space. Show that if $F : X \longrightarrow X$ is a shrinking map, meaning $d(F(x), F(y)) < d(x, y)$ for every $x \neq y \in X$, then F must have a fixed point. Show, moreover, that the fixed point of F is unique.

5. (a + b = 25 marks) a) What condition on a topological space X guarantees that the fundamental group $\pi_1(X, x_0)$ is independent of base point $x_0 \in X$? Prove it.

b) Let $p : E \longrightarrow B$ be a covering map, with E path connected. Show that if B is simply connected, then p must be a homeomorphism.

6. (25 marks) A Platonic solid is a polyhedron in which each face has the same number a of edges, and each vertex is attached to the same number b of edges.

a) Prove there are no more than five Platonic solids.

b) Find and prove a formula relating the Euler characteristic of a connected sum $A\#B$ to the Euler characteristics of two surfaces A and B.

7. (a + b = 25 marks) a) State the Classification Theorem for Compact Surfaces.

b) Where does a connected sum of a Klein bottle and a torus fit into this list? Prove it.