Introductory Topology Notes

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1 Cardinality and the Axiom of Choice

Definition 1.1 Two sets, A and B, have the *same cardinality* if and only if there exists a 1–1, onto function $f : A \to B$. The cardinality of a set A is denoted $|A|$. Let N be the set of positive integers, Z the set of all integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}^1$ the set of real numbers. We will accept as a fact that every non-empty set in N has a least element. A set X is finite if and only if there is a 1–1, onto function $f : X \to \{1, 2, \ldots, n\}$ where n is an element of N.

Theorem 1.1 The even positive integers have the same cardinality as the natural numbers.

Theorem 1.2 $|N| = |Z|$.

Theorem 1.3 Every subset of N is either finite or has the same cardinality as N.

Definition 1.2 A set which is finite or has the same cardinality as N is *countable* or has *countable cardinality*.

Theorem 1.4 $\mathbb{Q}$ is countable.

Theorem 1.5 The union of two countable sets is countable.

Theorem 1.6 The union of countably many countable sets is countable.

Theorem 1.7 The set of all finite subsets of a countable set is countable.

Definition 1.3 For any set A, 2^A denotes the set of all subsets of A. (The empty set, denoted $\emptyset$, is a subset of any set.) 2^A is called the *power set* of A.

Theorem 1.8 For any set A, there is a 1–1 function f from A into 2^A.

Theorem 1.9 For a set A, let P be the set of all functions from A to the two point set $\{0, 1\}$. Then $|P| = |2^A|$.

Theorem 1.10 There is a 1-1 correspondence between $2^{\mathbb{N}}$ and infinite sequences of 0's and 1's.

Theorem 1.11 (Cantor). There is no function from a set A onto 2^A.

Note that Theorem 11 implies that $2^{\mathbb{N}}$ is uncountable and that there are infinitely many different infinite cardinal numbers.

Theorem 1.12 A set is infinite if and only if there is a one-to-one function from the set into a proper subset of itself.

Theorem 1.13 There is a 1–1, onto function $f : [0, 1] \to [0, 1)$.

Theorem 1.14 (Schroeder-Bernstein). If A and B are sets such that there exist one-to-one functions f from A into B and g from B into A, then $|A| = |B|$.

(Note: We need to produce a 1-1, onto function $h : A \to B$. When defining h, for each point $x \in A$, either $h(x) = f(x)$ or $h(x) = g^{-1}(x)$. For some points x in A, you could not use g^{-1}. Start thinking about those points in beginning to define h.)

Theorem 1.15 $|\mathbb{R}^1| = |(0, 1)| = |[0, 1]|$.

Theorem 1.16 There is a 1-1 function from $\mathbb{R}^1 \to 2^{\mathbb{N}}$.

Theorem 1.17 $|\mathbb{R}^1| = |2^{\mathbb{N}}|$.

Below are listed Zorn's Lemma, the Axiom of Choice, and the Well-Ordering Principle. These three statements are equivalent and are used freely in most standard mathematics. We will use them freely in this course.

Definition 1.4 1. A set X is *partially ordered* by the relation $\leq$ if and only if, for any elements x, y, and z in X,

(a) if $x \leq y$ and $y \leq z$, then $x \leq z$, and

(b) if $x \leq y$ and $y \leq x$, then $x = y$.

2. Let X be a set partially ordered by $\leq$. Then an element m in X is a *maximal element* if and only if for any x in X, $m \leq x$ implies that $m = x$.

3. A set is *totally ordered* if and only if it is partially ordered and every two elements are comparable.

4. A set is *well-ordered* if and only if it is totally ordered and every non-empty subset has a least element.

Notice that any subset of a well-ordered set is well-ordered by the same ordering restricted to the subset.

Theorem 1.18 $\mathbb{R}^1$ with the usual ordering is totally ordered, but not well-ordered. N is well-ordered.

Example 1.1 For any set A, the set 2^A is partially ordered by set inclusion. The set A is a maximal element, and, in fact, the only maximal element in this ordering.

Zorn's Lemma. Let X be a partially ordered set in which each totally ordered subset has an upper bound in X. Then X has a maximal element.

Axiom of Choice. Let $\{A_\alpha\}_{\alpha\in\lambda}$ be a collection of non-empty sets. Then there is a function $f : \lambda \to \bigcup_{\alpha\in\lambda} A_\alpha$ such that for each α in λ, $f(\alpha)$ is an element of A_α.

Well-ordering Principle. Every set can be well-ordered. That is, every set is in 1–1 correspondence with a well-ordered set.

Ordinal numbers. The ordinal numbers with which we are most familiar are $0, 1, 2, 3, \ldots$. We can define ordinals in a manner which allows us to produce an ordered set of ordinals which includes infinite ordinals. We start with the empty set, $\emptyset$. This set corresponds to 0. The next ordinal, corresponding to 1, is the set containing the empty set, $\{\emptyset\}$. The next ordinal, corresponding to 2, is the set of its predecessors, namely $\{\emptyset, \{\emptyset\}\}$. The next ordinal, corresponding to 3, is the set of its predecessors, $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$.

Continuing in this fashion, we can define each subsequent ordinal as the set of its predecessors. For example, the first infinite ordinal, called ω_0, is the set $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \ldots\}$. The next ordinal is called $\omega_0 + 1$, then $\omega_0 + 2$, $\omega_0 + 3, \ldots$, then $2\omega_0, 2\omega_0 + 1, \ldots; \ldots k\omega_0, k\omega_0 + 1, k\omega_0 + 2, \ldots$, etc. Note that every ordinal number has an immediate successor; however, not every ordinal has an immediate predecessor. For example, ω_0 has no immediate predecessor. Note also that each ordinal is a set and, consequently, has a cardinality. The ordinal ω_0 is the first infinite ordinal and has the same cardinality as $\mathbb{N}$. $\omega_0 + 1$ also has countable cardinality, as do many others.

The first uncountable ordinal is called ω_1. Every ordinal preceding it is countable. The cardinality of ω_1 is less than or equal to the cardinality of $2^{\mathbb{N}}$. However, the Continuum Hypothesis below can be neither proved nor disproved — it is independent of the Axioms of Set Theory.

Continuum Hypothesis. The real numbers have the cardinality as ω_1.

Ordinal numbers are well-ordered, because the intersection of any set of ordinals is the smallest ordinal in the set, so every non-empty subset has a smallest element.

Theorem 1.19 Let $\{\alpha_i\}_{i \in \omega_0}$ be a countable set of ordinal numbers where each $\alpha_i < \omega_1$. Then there is an ordinal β such that $\alpha_i < \beta$ for each i and $\beta < \omega_1$.

Theorem 1.20 Let $\{\alpha_i\}_{i \in \omega_0}$ be an infinite set of ordinals. Then there is an ordinal β such that for every $\gamma < \beta$, there exists an α_i with $\gamma < \alpha_i < \beta$.