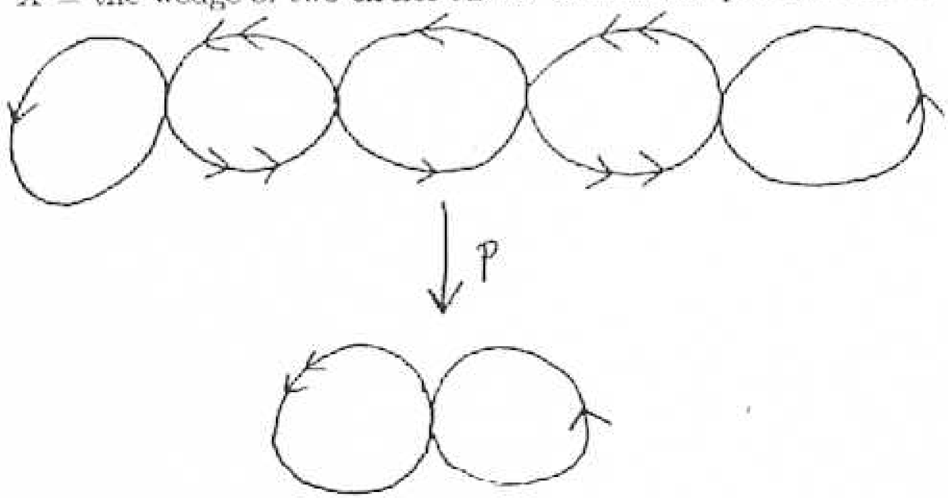
Covering Spaces

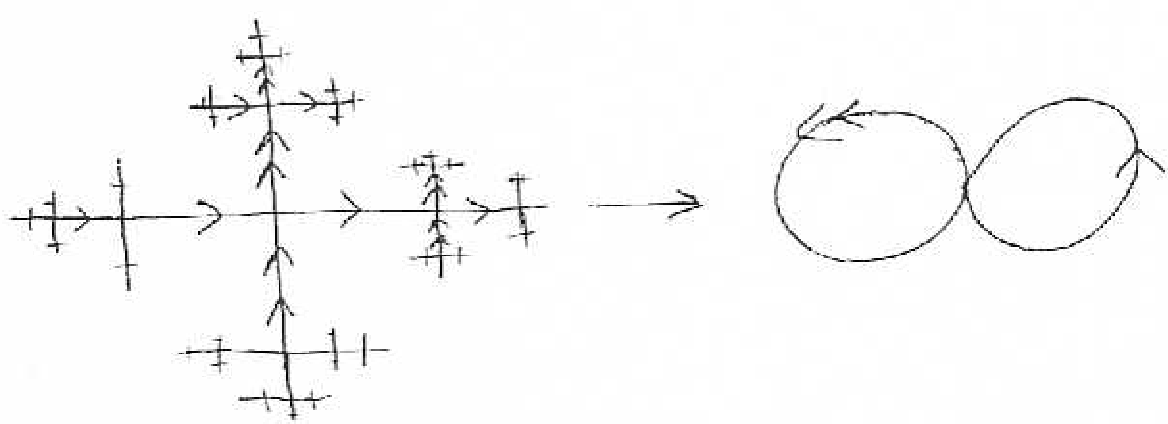
Definition. Let $X, \widetilde{X}$ be connected, locally arc-wise connected spaces, and $p : \widetilde{X} \to X$ a continuous function. Then the pair $(\widetilde{X}, p)$ is a *covering space of* X iff for all $x \in X$ there exists an open set U containing x such that p restricted to each component of $p^{-1}(U)$ is a homeomorphism onto U.

Example. Let $X = S^1$, $\widetilde{X} = \mathbb{R}^1$, and $p : \mathbb{R}^1 \to S^1$ be defined by $p(t) = (\cos t, \sin t)$.

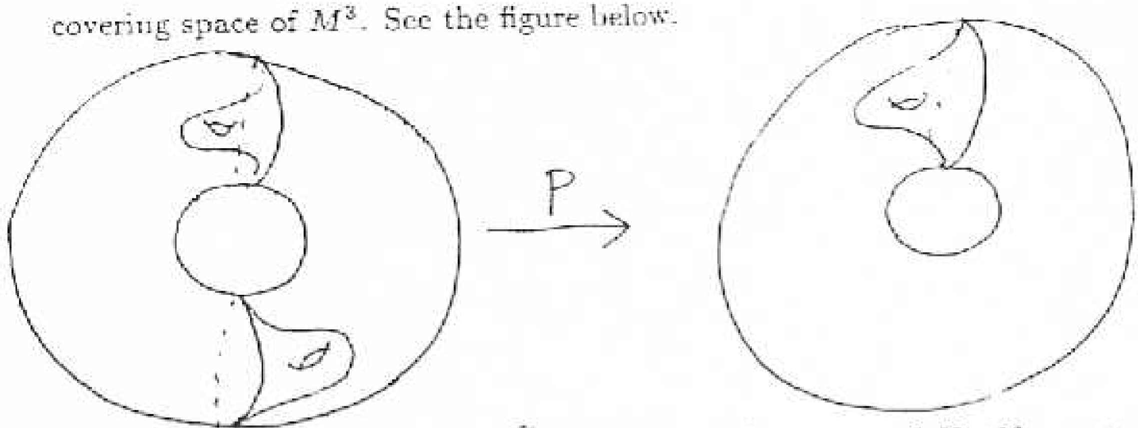
Example. $X \cong$ the wedge of two circles and $\widetilde{X}$ is as in the picture below.



Example.



Example. Let Σ^2 be a PL non-separating, two-sided, properly embedded surface in a connected 3-manifold M^3. Gluing two copies of $M^3 - N(\Sigma^2)$ together gives a covering space of M^3. See the figure below.



Theorem CS.1. Let $(\tilde{X}, p)$ be a covering space of X. If $x, y \in X$, then $|p^{-1}(x)| = |p^{-1}(y)|$.

Definition. If $(\widetilde{X}, p)$ is a covering space of a space X and $n = |p^{-1}(x)|$ for some $x \in X$, then $(\widetilde{X}, p)$ is called an *n-fold* cover.

Exercise. Find two 2-fold covers of the Klein bottle.

Definition. Given a covering space $(\widetilde{X}, p)$ of X and a continuous function $f : Y \to X$, then a continuous function $\tilde{f} : Y \to \widetilde{X}$ is called a *lift of* f if $p \circ \tilde{f} = f$.

Theorem CS.2. If $(\widetilde{X}, p)$ is a cover of X, Y is connected, and $f, g : Y \to \widetilde{X}$ are continuous functions such that $p \circ f = p \circ g$, then $\{y \mid f(y) = g(y)\}$ is empty or all of Y.

Theorem CS.3. Let $(\widetilde{X}, p)$ be a cover of X and let f be a path in X. Then for each $x_0 \in \widetilde{X}$ such that $p(x_0) = f(0)$, there exists a unique lift $\tilde{f}$ of f satisfying $\tilde{f}(0) = x_0$.

Theorem CS.4. (Homotopy Lifting Lemma). Let $(\widetilde{X}, p)$ be a cover of X and α, β two paths in X. If $\tilde{\alpha}, \tilde{\beta}$ are lifts of α, β satisfying $\tilde{\alpha}(0) = \tilde{\beta}(0)$, then $\tilde{\alpha} \sim \tilde{\beta}$ iff $\alpha \sim \beta$.

Theorem CS.5. Let $(\widetilde{X}, p)$ be a cover of X, α a loop in X, and $\tilde{x}_0 \in \widetilde{X}$ such that $p(\tilde{x}_0) = \alpha(0)$. Then α lifts to a loop based at $\tilde{x}_0$ iff $[\alpha] \in p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$.

Exercise. $\pi_1(S^1) \cong \mathbb{Z}$.

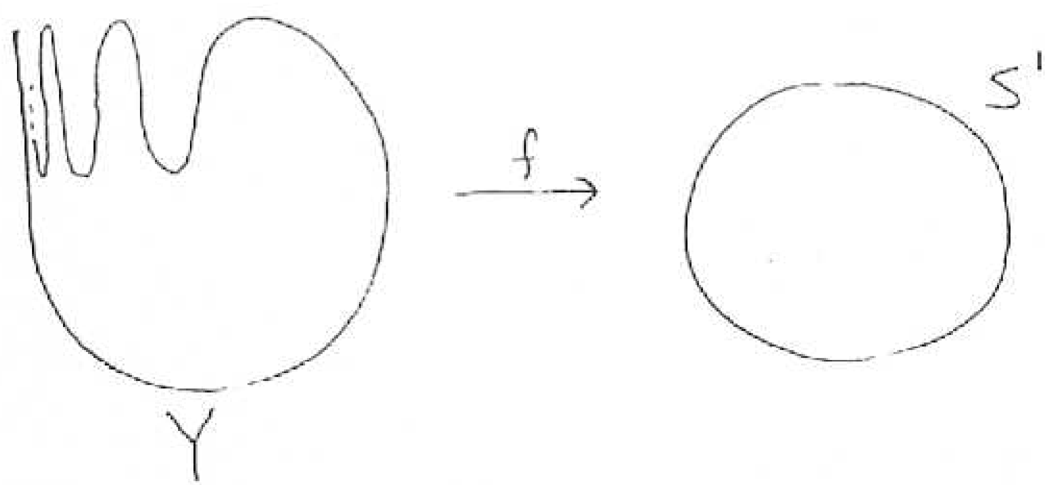
Theorem CS.6. If $(\tilde{X}, p)$ is a cover of X, then p_* is a monomorphism (*i.e.*, 1–1) from $\pi_1(\tilde{X})$ into $\pi_1(X)$.

Theorem CS.7. Let $(\tilde{X}, p)$ be a cover of X, $x_0 \in X$, and $\tilde{X}$ be path connected. Fix $\tilde{x}_0 \in p^{-1}(x_0)$. Then a subgroup H of $\pi_1(X)$ is in $\{p_*(\pi_1(\tilde{X}, \tilde{x}))\}_{p(\tilde{x})=x_0}$ iff H is a conjugate of $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

Theorem CS.8. Let $(\tilde{X}, p)$ be a cover of X and let $\tilde{X}$ be path connected. Choose $x \in X$, then $|p^{-1}(x)| = [\pi_1(x) : p_*(\pi_1(\tilde{x}))]$.

Example. Find a 3-fold cover $(\tilde{X}, p)$ of S^1 and describe $p_*(\pi_1(\tilde{X}))$.

Exercise. Let $X = S^1$, $\tilde{X} = \mathbb{R}$, $(\tilde{X}, p)$ be a covering space of X, and Y as in the figure. When does a map $f : Y \to X$ not have a lift?



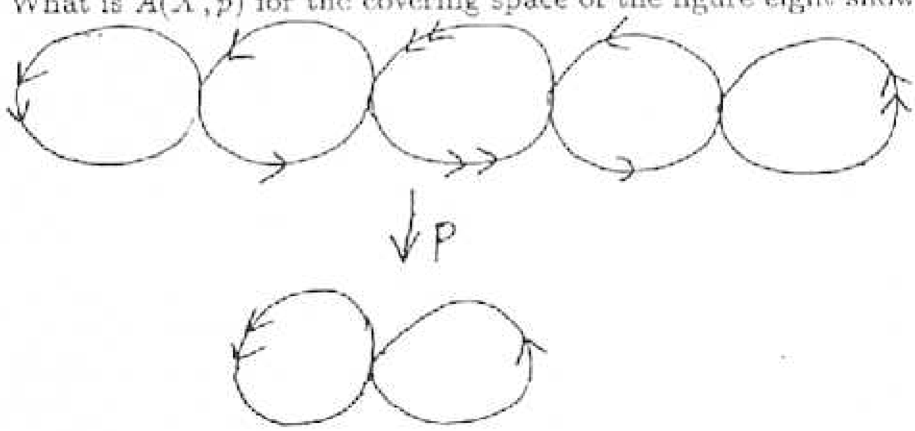
Theorem CS.9. Let $(\tilde{X}, p)$ be a covering space of X and $\tilde{x}_0 \in \tilde{X}$, $x_0 \in X$ with $p(\tilde{x}_0) = x_0$. Also let $f : Y \to X$ be continuous where Y is connected and locally arc-wise connected and $y_0 \in Y$ such that $f(y_0) = x_0$. Then there is a lift $\tilde{f} : Y \to \tilde{X}$ such that $p \circ \tilde{f} = f$ and $\tilde{f}(y_0) = \tilde{x}_0$ iff $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Furthermore, $\tilde{f}$ is unique.

Definition. Let $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ be covering spaces of X. Then a map $f : \tilde{X}_1 \to \tilde{X}_2$ such that $p_2 \circ f = p_1$ is called a *homomorphism*. If there exists a homomorphism $g : \widetilde{X}_2 \to \widetilde{X}_1$ such that f and g are inverses, then f is called an *isomorphism*.

Theorem CS.10. Let $(\widetilde{X}_1, p_1)$ and $(\widetilde{X}_2, p_2)$ be covering spaces of X. Let $\tilde{x}_1 \in \widetilde{X}_1$ and $\tilde{x}_2 \in X_2$ such that $p_1(x_1) = p_2(x_2)$. Then there is an isomorphism $f : \widetilde{X}_1 \to \widetilde{X}_2$ with $f(\widetilde{X}_1) = \widetilde{X}_2$ iff $p_*(\pi_1(\widetilde{X}_1, \tilde{x}_1)) = p_*(\pi_1(\widetilde{X}_2, \tilde{x}_2))$.

Definition. Let $(\widetilde{X}, p)$ be a covering space. Then an isomorphism from $\widetilde{X}$ to itself is called a *covering transformation*. The group of covering transformations is denoted $A(\widetilde{X}, p)$.

Exercise. What is $A(\widetilde{X}, p)$ for the covering space of the figure eight shown below.



Theorem CS.11. If $(\widetilde{X}, p)$ is a covering space of X and $f \in A(\widetilde{X}, p)$, then $f = Id_{\widetilde{X}}$ iff f has a fixed point.

Definition. Let $(\widetilde{X}, p)$ be a covering space of X. If $p_*(\pi_1(\widetilde{X})) \triangleleft \pi_1(X)$, then $(\widetilde{X}, p)$ is a *regular* covering space.

Exercise. Find all regular 4-fold covering spaces of a figure eight.

Theorem CS.12. If $(\widetilde{X}, p)$ is a regular cover of X and $x_1, x_2 \in \widetilde{X}$ such that $p(x_1) = p(x_2)$, then there exists a unique $h \in A(\widetilde{X}, p)$ such that $h(x_1) = x_2$.

Theorem CS.13. If $(\widetilde{X}, p)$ is a regular covering space of x and f is a homomorphism from $\widetilde{X}$ to $\widetilde{X}$, then $f \in A(\widetilde{X}, p)$.

Theorem CS.14. Let $(\widetilde{X}, p)$ be a regular covering space of X. Then $A(\widetilde{X}, p) \cong \pi_1(x)/p_*(\pi_1(\widetilde{X}))$.

Definition. A space X is called *semi-locally simply connected* if every $x \in X$ is contained in an open set U such that every loop in U based at x is homotopically trivial in X.

Theorem CS.15. Let X be connected, locally arc-wise connected, and semi-locally simply connected. Then for every $G < \pi_1(X, x_0)$ there is a connected covering space $(\widetilde{X}, p)$ of X and $\tilde{x}_0 \in \widetilde{X}$ such that $p_*(\pi_1(\widetilde{X}, \tilde{x}_0)) = G$. Furthermore, $(\widetilde{X}, p)$ is unique up to isomorphism.

Definition. A connected, locally arc-wise connected cover is called *universal* if its fundamental group is trivial.

Theorem CS.16. Every connected, locally arc-wise connected, semi-locally simply connected space has a universal covering space.

Exercise. Find universal covers for the Klein bottle, torus, and projective plane. Then show explicitly that $A(\widetilde{X}, p) \cong \pi_1(X)$.