2 General Topology

Definition 2.1 1. Suppose X is a set. Then $\mathcal{T}$ is a *topology* for X if and only if $\mathcal{T}$ is a collection of subsets of X such that

(a) $\emptyset \in \mathcal{T}$,

(b) $X \in \mathcal{T}$,

(c) if $A \in \mathcal{T}$ and $B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$,

(d) if $\{A_\alpha\}_{\alpha \in \lambda}$ is any collection of sets each of which is in $\mathcal{T}$, then $\bigcup_{\alpha \in \lambda} A_\alpha \in \mathcal{T}$.

2. A *topological space* is an ordered pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a topology for X.

3. If $(X, \mathcal{T})$ is a topological space, then U is an *open set* in $(X, \mathcal{T})$ if and only if $U \in \mathcal{T}$.

Several examples of topological spaces are listed below.

Example 2.1 For a set X, let 2^X be the set of all subsets of X. Then 2^X is called the *discrete topology* on X. The space $(X, 2^X)$ is called a *discrete topological space*.

Example 2.2 For a set X, $\{\emptyset, X\}$ is called the *indiscrete topology* for X. So $(X, \{\emptyset, X\})$ is an indiscrete topological space.

Example 2.3 For any set X, the *finite complement topology* for X is described as follows: a subset U of X is open if and only if $U = \emptyset$ or $X - U$ is finite.

Example 2.4 Let $\mathbb{R}^n$ be the set of all n-tuples of real numbers. We will define the distance $d(x, y)$ between points $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ by the equation

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}.$$

A topology $\mathcal{T}$ for $\mathbb{R}^n$ is defined as follows: a subset U of $\mathbb{R}^n$ belongs to $\mathcal{T}$ if and only if for each point p of U there is a positive number ε so that $\{x \mid d(p, x) < \varepsilon\}$ is a subset of U. This topology $\mathcal{T}$ is called the *usual topology for* $\mathbb{R}^n$.

Theorem 2.1 Let $\{U_i\}_{i=1}^n$ be a finite collection of open sets in a topological space $(X, \mathcal{T})$. Then $\bigcap_{i=1}^n U_i$ is open.

Give an example to show that an infinite intersection of open sets need not be open.

Definition 2.2 . Let $(X, \mathcal{T})$ be a topological space, A be a subset of X, and p be a point in X. Then:

1. p is a *limit point of* A if and only if for each open set U containing p, $(U - \{p\}) \cap A \neq \emptyset$. Notice that p may or may not belong to A.
2. If $p \in A$ but p is not a limit point of A, then p is an *isolated point* of A.
3. The *closure* of A (denoted $\overline{A}$ or $\mathrm{Cl}(A)$) is A together with all limit points of A.
4. The set A is *closed* iff A contains all its limit points, *i.e.*, $\overline{A} = A$.

Theorem 2.2 Suppose $p \notin A$ in a topological space $(X, \mathcal{T})$. Then p is not a limit point of A if and only if there exists an open set U with $p \in U$ and $U \cap A = \emptyset$.

Theorem 2.3 For any topological space $(X, \mathcal{T})$ and subset A of X, $\overline{A}$ is closed.

Theorem 2.4 Let X be a topological space. Then a subset A of X is closed if and only if $X - A$ is open.

Theorem 2.5 Let X be a topological space. Let U be open and A be closed subsets of X. Then $U - A$ is open.

Theorem 2.6 The union of finitely many closed sets in a topological space is closed.

Give an example to show that a union of infinitely many closed sets may not be closed.

Theorem 2.7 Let $\{A_\alpha\}_{\alpha \in \lambda}$ be a collection of closed subsets of a topological space X. Then $\bigcap_{\alpha \in \lambda} A_\alpha$ is closed.

Theorem 2.8 Suppose A is a subset of X, a topological space. Then $\overline{A}$ equals the intersection of all closed sets containing A.

Theorem 2.9 Let A, B be subsets of a topological space X. Then

1. if $A \subset B$, then $\overline{A} \subset \overline{B}$; and
2. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Definition 2.3 Let $\mathcal{T}$ be a topology on a set X and let $\mathcal{B}$ be a subset of $\mathcal{T}$. Then $\mathcal{B}$ is a *basis* for the topology $\mathcal{T}$ if and only if every element of $\mathcal{T}$ is the union of elements in $\mathcal{B}$.

Theorem 2.10 Let $(X, \mathcal{T})$ be a topological space and $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is a basis for $\mathcal{T}$ if and only if $\mathcal{B} \subset \mathcal{T}$, $\phi \in \mathcal{B}$, and for each set U in $\mathcal{T}$ and point p in U there is a set V in $\mathcal{B}$ such that $p \in V \subset U$.

Theorem 2.11 Let $\mathcal{B}_1 = \{(a, b) \subset \mathbb{R}^1 \mid a \text{ and } b \text{ are rational numbers}\}$. Then $\mathcal{B}_1$ is a basis for the usual topology on $\mathbb{R}^1$. Let $\mathcal{B}_2 = \{(a, b) \cup (c, d) \subset \mathbb{R}^1 \mid a, b, c \text{ and } d \text{ are irrational numbers}\}$. Then $\mathcal{B}_2$ is also a basis for the usual topology on $\mathbb{R}^1$.

Suppose you are given a set X and a collection $\mathcal{B}$ of subsets of X. Under what circumstances is $\mathcal{B}$ a basis for a topology on X? This question is answered in the following theorem.

Theorem 2.12 Suppose X is a set and $\mathcal{B}$ is a collection of subsets of X. Then $\mathcal{B}$ is a basis for a topology for X if and only if the following conditions hold.

1. $\emptyset \in \mathcal{B}$
2. for each point p in X there is a set U in $\mathcal{B}$ with $p \in U$, and
3. if U and V are sets in $\mathcal{B}$ and p is a point in $U \cap V$, there is a set W in $\mathcal{B}$ so that $p \in W \subset (U \cap V)$.

Theorem 12 allows one to describe topological spaces by first specifying a set X and then a collection $\mathcal{B}$ of subsets of X which satisfy the conditions of Theorem 12. The topology $\mathcal{T}$ whose basis is $\mathcal{B}$ is thereby described.

Example 2.5 $\mathbb{R}^1$ (bad). The points of $\mathbb{R}^1$ (bad) are the reals. A basis for the topology of $\mathbb{R}^1$ (bad) consists of all sets of the form $[a,b) = \{x \in \mathbb{R}^1 \mid a \leq x < b\}$.

Show that every open set in the usual topology on $\mathbb{R}^1$ is open in $\mathbb{R}^1$ (bad).

Definition 2.4 Suppose X is a set. A function d from $X \times X$ into $\mathbb{R}^1_+$, the non-negative reals, is a *metric* for X if and only if the following conditions are satisfied.

1. $d(x,y) = 0$ if and only if $x = y$
2. $d(x,y) = d(y,x)$, and
3. $d(x,z) \leq d(x,y) + d(y,z)$.

If d is a metric for X, then $d(x,y)$ is called the distance from x to y.

Suppose X is a set, d is a metric for X, $p \in X$, and $\varepsilon \in \mathbb{R}^1_+$. Then the *open* ε *ball about* p is defined by $B_\varepsilon(p) = \{x \in X \mid d(x,p) < \varepsilon\}$. The *d-metric topology* for X is the topology whose basis is all the $B_\varepsilon(p)$'s.

Theorem 2.13 Let d be a metric on a space X. Then the collection of all open ε balls is a basis.

Now suppose that $(X, \mathcal{T})$ is a topological space. Then $(X, \mathcal{T})$ is a *metric space* (or *metrizable*) iff there is a metric d on X for which $\mathcal{T}$ is the d-metric topology. If X is a metric space, then the statement that *d is a metric for X* means that the d-metric topology is the topology for X.

Notice that different metrics may generate the same topology. As an exercise find several metrics for $\mathbb{R}^n$.

Theorem 2.14 Let X be a metric space and let $a > 0$. Then $B = \{B_\varepsilon(p) \mid p \in X, \varepsilon < a\}$ is a basis for the d-metric topology on X.

Theorem 2.15 If X is a metric space with topology $\mathcal{T}$, then there is a metric d for X that generates $\mathcal{T}$ such that for each $x, y \in X$, $d(x,y) < 1$.

Example 2.6 Let X be a set totally ordered by $<$. Let $\mathcal{B}$ be the collection of all subsets of X of one of the following three forms: $\{x \in X \mid x < a$ for some $a \in X\}$, $\{x \in X \mid a < x$ for some $a \in X\}$, or $\{x \in X \mid a < x < b$ for some $a, b \in X\}$. Then $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on X. The topology $\mathcal{T}$ is called the *order topology* for X.

Example 2.7 The usual topology $\mathbb{R}^1$ is the order topology given by the usual order.

Example 2.8 For each ordinal α, the predecessors of α with the order topology form a space called α.

Definition 2.5 Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{S}$ be a collection of subsets of X. Then $\mathcal{S}$ is a *sub-basis* of $\mathcal{T}$ if and only if the collection $\mathcal{B}$ of all finite intersections of sets in $\mathcal{S}$ is a basis for $\mathcal{T}$.

Theorem 2.16 A basis for a topology is also a subbasis.

Theorem 2.17 Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{S}$ be a collection of subsets of X. Then $\mathcal{S}$ is a sub-basis for $\mathcal{T}$ if and only if each element of $\mathcal{S}$ is in $\mathcal{T}$, there is a finite collection $\{V_i\}_{i=1}^n$ of elements of $\mathcal{S}$ such that $\bigcap_{i=1}^n V_i = \emptyset$, and for each set U in $\mathcal{T}$ and point p in U there is a finite collection $\{V_i\}_{i=1}^n$ of elements of $\mathcal{S}$ such that

$$p \in \bigcap_{i=1}^{n} V_i \quad \text{and} \quad \bigcap_{i=1}^{n} V_i \subset U .$$

Theorem 2.18 Let $\mathcal{S}$ be the collection of all subsets of $\mathbb{R}^1$ of one of the following two forms: $\{x \mid x < a \text{ for some } a \in \mathbb{R}^1\}$ and $\{x \mid a < x \text{ for some } a \in \mathbb{R}^1\}$. Then $\mathcal{S}$ is a sub-basis for $\mathbb{R}^1$ with the usual topology.

We seek to answer the question of when a given collection $\mathcal{S}$ of subsets of a set X is a sub-basis for a topology on X.

Theorem 2.19 Let $\mathcal{S}$ be a collection of subsets of a set X. Then $\mathcal{S}$ is a sub-basis for a topology on X if and only if every point of X is in some element of $\mathcal{S}$ and there are sets $\{U_i\}_{i=1}^n$ in $\mathcal{S}$ so that

$$\bigcap_{i=1}^{n} U_i = \emptyset .$$

Theorem 19 can be used to describe topologies by presenting a sub-basis for them.

Example 2.9 Let 2^X be the set of all functions from the set X into the two point set $\{0, 1\}$. Let $\mathcal{S}$ be the collection of all subsets of 2^X of the form $U(x, \varepsilon) = \{f \in 2^X \mid f(x) = \varepsilon\}$ where $\varepsilon = 0$ or 1. Let $\mathcal{T}$ be the topology on 2^X with sub-basis $\mathcal{S}$. (This topology is really the product topology, but we will not give a general definition of product topology until later.)

Question 2.1 *Under what conditions are two basic open sets in Example 9 disjoint?*

Theorem 2.20 Suppose $(X, \mathcal{T})$ is a topological space, $Y \subset X$, and $\mathcal{T}_Y = \{U \mid \text{for some } V \text{ in } \mathcal{T}, U = V \cap Y\}$. Then $\mathcal{T}_Y$ is a topology for Y.

Theorem 20 allows us to define a topology on a subset Y of X when $(X, \mathcal{T})$ is a topological space. The topology $\mathcal{T}_Y$ of Y of Theorem 20 is called the *relative topology* or *subspace topology*. The topological space $(Y, \mathcal{S})$ is a *subspace* of $(X, \mathcal{T})$ if and only if Y is a subset of X and $\mathcal{S}$ is the relative topology on Y.

Theorem 2.21 If X is a metric space and $Y \subset X$, then Y is a metric space.