3 Separation Properties

Definition 3.1 . Let $(X, \mathcal{T})$ be a topological space:

1. X is T_1 if and only if every point in X is a closed set.
2. X is *Hausdorff* (or T_2) if and only if for each pair of points x, y in X, there are disjoint open sets U and V in $\mathcal{T}$ such that $x \in U$ and $y \in V$.
3. X is *regular* if and only if for each $x \in X$ and closed set A in X with $x \notin A$, there are open sets U, V such that $x \in U$, $A \subset V$ and $U \cap V = \emptyset$.
4. X is *normal* if and only if for each pair of disjoint closed sets A and B in X, there are open sets U, V such that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

Theorem 3.1 Every Hausdorff space is T_1.

Theorem 3.2 Every regular, T_1 space is Hausdorff.

Theorem 3.3 Every normal, T_1 space is regular.

Theorem 3.4 A topological space X is regular if and only if for each point p in X and open set U containing p there is an open set V such that $p \in V$ and $\overline{V} \subset U$.

Theorem 3.5 A topological space X is normal if and only if for each closed set A in X and open set U containing A there is an open set V such that $A \subset V$, and $\overline{V} \subset U$.

Theorem 3.6 A topological space X is normal if and only if for each pair of disjoint closed sets A and B, there are disjoint open sets U and V such that $A \subset U$, $B \subset V$, and $\overline{U} \cap \overline{V} = \emptyset$.

Exercise 3.1 Find two disjoint closed subsets A and B of a metric space such that
$\inf\{d(a, b) \mid a \in A \text{ and } b \in B\} = 0$.

Theorem 3.7 A metric space is normal.

Definition 3.2 Let P be a topological property (such as T_1, Hausdorff, etc.). A topological space X is *hereditarily* P if and only if for each subspace Y of X, Y has property P.

Theorem 3.8 A Hausdorff space is hereditarily Hausdorff.

Theorem 3.9 A regular space is hereditarily regular.

Theorem 3.10 Let Y be a subset of X and let A be a subset of Y. Then the closure of A in the relative topology on Y equals the intersection of Y with the closure of A in X.

Theorem 3.11 Let A be a closed subset of a normal space X. Then A is normal when given the relative topology.

Theorem 3.12 Normality Lemma Let A and B be subsets of a topological space X and let $\{U_i\}_{i\in\omega_0}$ and $\{V_i\}_{i\in\omega_0}$ be two collections of open sets such that

1. $A \subset \bigcup_{i\in\omega_0} U_i$,
2. $B \subset \bigcup_{i\in\omega_0} V_i$,
3. for each i in ω_0, $\overline{U}_i \cap B = \emptyset$ and $\overline{V}_i \cap A = \emptyset$.

Then there are open sets U and V such that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.