4 Countability Properties

Definition 4.1 Let A be a subset of a topological space X. Then A is *dense* in X if and only if $\overline{A} = X$.

1. A space X is *separable* if and only if X has a countable dense subset.
2. A space X is *2nd countable* if and only if X has a countable basis.
3. Let p be a point in a space X. A collection of open sets $\{U_\alpha\}_{\alpha\in\lambda}$ in X is a *neighborhood basis for* p if and only if $p \in U_\alpha$ for each $\alpha \in \lambda$ and for every open set U in X with p in U, there is an α in λ such that $U_\alpha \subset U$.
4. A space X is *1st countable* if and only if for each point x in X, x has a neighborhood basis consisting of a countable number of sets.
5. A space X has the *Souslin property* if and only if X does *not* contain an uncountable collection of disjoint open sets.

Theorem 4.1 A 2nd countable space is separable.

Theorem 4.2 A 2nd countable space is 1st countable.

Theorem 4.3 A 2nd countable space is hereditarily 2nd countable.

Theorem 4.4 A separable space has the Souslin property.

Theorem 4.5 If X is a separable, Hausdorff space, then $|X| \leq |2^{2^\omega}|$.

Theorem 4.6 For any separable space X, the topological space 2^X has the Souslin property.

Theorem 4.7 The space $2^{\mathbb{R}^1}$ is separable.

Definition 4.2 Let $P = \{p_i\}_{i\in\omega_0}$ be a sequence of points in a space X. Then the sequence P *converges to* a point x if and only if for every open set U containing x there is an integer M such that for each $m > M$, $p_m \in U$.

Theorem 4.8 If $p \in X$ and p has a countable neighborhood basis, then p has a nested countable neighborhood basis.

Theorem 4.9 Suppose x is a limit point of the set A in a 1st countable space X. Then there is a sequence of points in A which converges to x.

Theorem 4.10 Every uncountable set in a 2nd countable space has a limit point.