5 Covering Properties

Definition 5.1 Let A be a subset of X and let $\mathcal{B} = \{B_\alpha\}_{\alpha\in\lambda}$ be a collection of subsets of X. Then $\mathcal{B}$ is a *cover of* A if and only if $A \subset \bigcup_{\alpha\in\lambda} B_\alpha$. $\mathcal{B}$ is an *open cover* if and only if each B_α is open.

Definition 5.2 A space X is *compact* if and only if every open cover $\mathcal{B}$ of X has a finite subcover $\mathcal{C}$. That is, $\mathcal{C}$ is a finite open cover of X each of whose elements is a set in $\mathcal{B}$.

Definition 5.3 A space X is *countably compact* if and only if every countable open cover of X has a finite subcover.

Definition 5.4 A space X is *Lindelöf* if and only if every open cover of X has a countable subcover.

Definition 5.5 A collection $\mathcal{B} = \{B_\alpha\}_{\alpha\in\lambda}$ of subsets of a space X is *locally finite* if and only if for each point p in X there is an open set U containing p such that U intersects only finitely many elements of $\mathcal{B}$.

Example 5.1 . Let $\mathcal{B} = \{[n, n+1] \subset \mathbb{R}^1 \mid n \text{ is an integer}\}$. Then $\mathcal{B}$ is a locally finite collection in $\mathbb{R}^1$ (usual).

Definition 5.6 Let $\mathcal{B} = \{B_\alpha\}_{\alpha\in\lambda}$ be a cover of X. Then $\mathcal{C} = \{C_\beta\}_{\beta\in\mu}$ is a *refinement* of $\mathcal{B}$ if and only if (i) $\mathcal{C}$ is a cover of X and (ii) for each $\beta \in \mu$ there is an $\alpha \in \lambda$ such that $C_\beta \subset B_\alpha$. The collection $\mathcal{C}$ is an *open refinement* if and only if each C_β is an open set.

Definition 5.7 A space X is *paracompact* if and only if every open cover of X has a locally finite open refinement and X is Hausdorff.

Theorem 5.1 Every countably compact and Lindelöf space is compact.

Theorem 5.2 Every 2nd countable space is Lindelöf.

Theorem 5.3 Every compact, Hausdorff space is paracompact.

Theorem 5.4 Let A be a closed subspace of a compact space (respectively, countably compact, Lindelöf, paracompact). Then A is compact (respectively, countably compact, Lindelöf, paracompact).

Theorem 5.5 Let $\mathcal{B}$ be a basis for a space X. Then X is compact (respectively Lindelöf) if and only if every cover of X by basic open sets has a finite (respectively countable) subcover.

Theorem 5.6 The closed subspace $[0,1]$ in the $\mathbb{R}^1$ (usual) topology is compact.

Theorem 5.7 Let A be a compact subspace of a Hausdorff space X. Then A is closed.

Theorem 5.8 (The Heine-Borel Theorem). Let A be a subset of $\mathbb{R}^1$ with the usual topology. Then A is compact if and only if A is closed and bounded.

Theorem 5.9 If X is a Lindelöf space, then every uncountable subset of X has a limit point.

Theorem 5.10 Let X be a T_1 space. Then X is countably compact if and only if every infinite subset of X has a limit point.

Theorem 5.11 ω_1 is countably compact but not compact.

Theorem 5.12 $\omega_1 + 1$ is compact.

Theorem 5.13 (The Alexander Sub-basis Theorem) Let $\mathcal{S}$ be a sub-basis for a space X. Then X is compact if and only if every sub-basic open cover has a finite subcover. (A sub-basic open cover is a cover of X each element of which is a set in the sub-basis.)

Theorem 5.14 2^X is compact.

Theorem 5.15 A compact, Hausdorff space is normal.

Theorem 5.16 A regular, Lindelöf space is normal.

Theorem 5.17 Let $\mathcal{B} = \{B_\alpha\}_{\alpha\in\lambda}$ be a locally finite collection of subsets of a space X. Let C be a subset of λ. Then $\mathrm{Cl}(\bigcup_{\alpha\in C} B_\alpha) = \bigcup_{\alpha\in C} \overline{B}_\alpha$.

Theorem 5.18 A paracompact space is normal.

Theorem 5.19 A regular, T_1, Lindelöf space is paracompact.

Theorem 5.20 A metric space is paracompact.

Theorem 5.21 Let $\{U_\alpha\}_{\alpha\in\lambda}$ be an open cover of a compact set A in a metric space X. Then there exists a $\delta > 0$ such that for every point p in A, $B_\delta(p) \subset U_\alpha$ for some α.

A δ satisfying the theorem above is called a *Lebesgue number*.

Theorem 5.22 In a metric space X, the following are equivalent:

1. X is 2nd countable,
2. X is separable,
3. X has the Souslin property,
4. X is Lindelöf,
5. every uncountable set in X has a limit point.

Exercise 5.1 Make a grid with all our examples down the side and all our separation, countability, and covering properties across the top. Fill in squares indicating what examples have what properties.