6 Continuity and homeomorphisms

Definition 6.1 Let X and Y be topological spaces. A function $f : X \to Y$ is a *continuous function* if and only if for every open set U in Y, $f^{-1}(U)$ is open in X.

Theorem 6.1 Let $f : X \to Y$ be a function. Then the following are equivalent:

1. f is continuous,
2. for every closed set K in Y, $f^{-1}(K)$ is closed in X,
3. if p is a limit point of A in X, then $f(p)$ belongs to $\mathrm{Cl}(f(A))$.

Theorem 6.2 Let X be a metric space and Y a topological space. Then a function $f : X \to Y$ is continuous if and only if for each convergent sequence $x_n \to x$, $f(x_n)$ converges to $f(x)$.

Theorem 6.3 Let X be a compact space and let $f : X \to Y$ be a continuous function that is onto. Then Y is compact.

Theorem 6.4 Let X be a separable space and let $f : X \to Y$ be a continuous, onto map. Then Y is separable.

Theorem 6.5 Let A and B be disjoint closed sets in a normal space X. Then for each diadic rational r (that is, r can be written as a quotient of integers with denominator a power of 2) there exists an open set U_r such that $A \subset U_0$, $B \subset (X - U_1)$, and for $r < s$, $\mathrm{Cl}(U_r) \subset U_s$.

Theorem 6.6 (Urysohn's Lemma) A space X is normal if and only if for each pair of disjoint closed sets A and B in X, there exists a continuous function $f : X \to [0, 1]$ such that $A \subset f^{-1}(0)$ and $B \subset f^{-1}(1)$.

Theorem 6.7 Let X be a normal space, A be a closed subset of X, and $f : A \to [0, 1]$ be a continuous function. Then for any r in $(0, 1)$, there is an open set U_r in X such that $f^{-1}([0, r)) = U_r \cap A$ and $\overline{U}_r \cap A \subset f^{-1}([0, r])$.

Theorem 6.8 (The Tietze Extension Theorem) A space X is normal if and only if every continuous function f from a closed set A in X into $[0, 1]$ can be extended to a continuous function $F : X \to [0, 1]$. (F extends f means for each point x in A, $F(x) = f(x)$.)

Theorem 6.9 (The Tietze Extension Theorem) A space X is normal if and only if every continuous function f from a closed set A in X into $(0,1)$ can be extended to a continuous function $F : X \to (0,1)$. ($\mathbb{R}^1$ could be substituted for $(0,1)$ in this theorem.)

Theorem 6.10 If X and Y are metric spaces with metrics d_X and d_Y respectively, then a function $f : X \to Y$ is continuous if and only if for each point x in X and $\varepsilon > 0$, there is a $\delta > 0$ such that for each $y \in X$ with $d_X(x,y) < \delta$, then $d_Y(f(x), f(y)) < \varepsilon$.

Definition 6.2 A function f from a metric space (X, d_X) to a metric space (Y, d_Y) is *uniformly continuous* if and only if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for every $x, y \in X$, if $d_X(x,y) < \delta$, then $d_Y(f(x), f(y)) < \varepsilon$.

Give an example of a continuous function from $\mathbb{R}^1$ to $\mathbb{R}^1$ which is not uniformly continuous.

Theorem 6.11 Let $f : X \to Y$ be a continuous function from a compact metric space to a metric space Y. Then f is uniformly continuous for any choice of metrics for X and Y.

Definition 6.3 . A sequence $\{a_i\}_{i\in\omega}$ in a metric space X is *Cauchy* if and only if for every $\varepsilon > 0$, there exists an N such that for every $i, j > N$, $d(a_i, a_j) < \varepsilon$.
Let d_X be a metric for a topology on a metric space X. (X, d_X) is complete if and only if every Cauchy sequence converges.

Theorem 6.12 Let $f_i : (X, d_X) \to (Y \text{ complete}, d_Y)$ $(i \in \omega)$ be a sequence of continuous functions such that for each $i \in \omega$, and point x in X, $d_Y(f_i(x), f_{i+1}(x)) < 1/2^i$. Then $\lim_{i\to\infty} f_i$ exists and is continuous.

Definition 6.4 A continuous function $f : X \to Y$ is *closed* (resp. *open*) if and only if for every closed (resp. open) set A in X, $f(A)$ is closed (resp. open) in Y.

Theorem 6.13 Let X be compact and Y Hausdorff. Then any continuous function $f : X \to Y$ is closed.

Definition 6.5 A function $f : X \to Y$ is a *homeomorphism* if and only if f is continuous, 1-1 and onto and $f^{-1} : Y \to X$ is also continuous.

Theorem 6.14 For a continuous function $f : X \to Y$, the following are equivalent:

a) f is a homeomorphism.

b) f is 1–1, onto and closed.

c) f is 1–1, onto and open.

Definition 6.6 Spaces X and Y are ***homeomorphic*** if and only if there is a homeomorphism $f : X \to Y$.

Theorem 6.15 For points $a < b$ in $\mathbb{R}^1$, the interval (a, b) is homeomorphic to $\mathbb{R}^1$.

Theorem 6.16 Suppose $f : X \to Y$ is a 1–1 and onto continuous function, X is compact and Y is Hausdorff. Then f is a homeomorphism.

Theorem 6.17 Let $f : X \to Y$ be a function. Suppose $X = A \cup B$ where A and B are closed subsets of X. If $f \mid A$ is continuous and $f \mid B$ is continuous, then f is continuous.