7 Products

Let $\{X_\alpha\}_{\alpha\in\lambda}$ be a collection of spaces. The *product* $\prod_{\alpha\in\lambda} X_\alpha$, or *Cartesian product*, is a generalization of the familiar n-tuples. Define $\prod_{\alpha\in\lambda} X_\alpha$ to be $\{f : \lambda \to \bigcup_{\alpha\in\lambda} X_\alpha \mid f(\alpha) \in X_\alpha\}$. So a point in $\prod_{\alpha\in\lambda} X_\alpha$ can be thought of as a function from the indexing set into $\bigcup_{\alpha\in\lambda} X_\alpha$. If $f \in \prod_{\alpha\in\lambda} X_\alpha$, $f(\alpha)$ is the α^{th} *coordinate* of f. We could write f as $\{f_\alpha\}_{\alpha\in\lambda}$ where $f(\alpha) = f_\alpha$.

For each β in λ, define the projection function $\pi_\beta : \prod_{\alpha\in\lambda} X_\alpha \to X_\beta$ by $\pi_\beta(f) = f(\beta)$. A subbasis for the *product topology* on $\prod_{\alpha\in\lambda} X_\alpha$ is the collection of all sets of the form $\pi_\beta^{-1}(U_\beta)$ where U_β is open in X_β. Why is it appropriate to refer to this topology as the finite gate topology?

Theorem 7.1 The space 2^X described before is really the product $\prod_{x\in X}\{0,1\}_x$.

Theorem 7.2 The function $\pi_\beta : \prod_{\alpha\in\lambda} X_\alpha \to X_\beta$ is a continuous, open, onto map.

Theorem 7.3 The function $\pi_\beta : \prod_{\alpha\in\lambda} X_\alpha \to X_\beta$ need not be closed.

Theorem 7.4 A function $g : Y \to \prod_{\alpha\in\lambda} X_\alpha$ is continuous if and only if $\pi_\beta \circ g$ is continuous for each β in λ.

Theorem 7.5 Let $\{X_i\}_{i\in\omega}$ be a countable collection of metric spaces. Then $\prod_{i\in\omega} X_i$ is a metric space.

Theorem 7.6 The space $\mathbb{R}^n$ is homeomorphic to $\prod_{i=1}^n \mathbb{R}^1_i$ where $\mathbb{R}^1_i = \mathbb{R}^1$.

Theorem 7.7 $\mathbb{R}^1$ (bad) is normal, but $\mathbb{R}^1$ (bad) $\times\, \mathbb{R}^1$ (bad) is not normal.

Theorem 7.8 Let $\{X_\beta\}_{\beta\in\mu}$ be a collection of Hausdorff (resp. regular) spaces. Then $\prod_{\beta\in\mu} X_\beta$ is Hausdorff (resp. regular).

Theorem 7.9 Let $\{X_\beta\}_{\beta\in\mu}$ be a collection of separable spaces where $|\mu| \leq 2^{\omega}$, then $\prod_{\beta\in\mu} X_\beta$ is separable.

Theorem 7.10 Let $\{X_\beta\}_{\beta\in\mu}$ be a collection of separable spaces. Then $\prod_{\beta\in\mu} X_\beta$ has the Souslin property.

Theorem 7.11 (The Tychonoff Theorem) Let $\{X_\beta\}_{\beta\in\mu}$ be a collection of compact spaces. Then $\prod_{\beta\in\mu} X_\beta$ is compact.

Definition 7.1 A space X is *completely regular* if and only if for each point p and open set U with $p \in U$, there is a continuous function $f : X \to [0, 1]$ such that $f(p) = 0$ and $f(X - U) = 1$.

Theorem 7.12 Let X be a completely regular, T_1 space. Then X can be embedded in a product of $[0, 1]$'s.