8 Connectedness

Definition 8.1 .

1. Subsets A, B of X are *separated* if and only if $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.

2. A space X is *connected* if and only if X is not the union of two non-empty separated sets. The notation $X = A \mid B$ means $X = A \cup B$ and A and B are separated sets.

Theorem 8.1 The following are equivalent:

1. X is connected;

2. there is no continuous function $f : X \to \mathbb{R}^1$ such that $f(X) = \{0, 1\}$;

3. X is not the union of two non-empty, disjoint open sets;

4. X is not the union of two non-empty, disjoint closed sets.

Theorem 8.2 The space $\mathbb{R}^1$ is connected.

Theorem 8.3 Let A, B be separated subsets of a space X. If C is a connected subset of $A \cup B$, then $C \subset A$, or $C \subset B$.

Theorem 8.4 Let C be a connected subset of X. If D is a subset of X so that $C \subset D \subset \overline{C}$, then D is connected.

Example 8.1 . Let

$$X = \Big\{(x, y) \in \mathbb{R}^2 \mid x = 0\,,\ y \in [-1, 1]\Big\} \bigcup \Big\{(x, y) \in \mathbb{R}^2 \mid x \in (0, 1]\,,\ y = \sin\frac{1}{x}\Big\}.$$

This example is the closure of the $\sin 1/x$ curve.

Theorem 8.5 The closure of the $\sin 1/x$ curve is connected.

Theorem 8.6 Let $\{C_\alpha\}_{\alpha\in\lambda}$ be a collection of connected subsets of X and E be another connected subset of X so that for each α in λ, $E \cap C_\alpha \neq \emptyset$. Then $E \bigcup (\bigcup_{\alpha\in\lambda} C_\alpha)$ is connected.

Theorem 8.7 Let $f : X \overset{\text{onto}}{\Rightarrow} Y$ be a continuous function. If X is connected, then Y is connected.

Theorem 8.8 For spaces X and Y, $X \times Y$ is connected if and only if each of X and Y is connected.

Theorem 8.9 For spaces $\{X_\alpha\}_{\alpha\in\lambda}$, $\prod_{\alpha\in\lambda} X_\alpha$ is connected if and only if for each α in λ, X_α is connected.

Theorem 8.10 Let A be a countable subset of $\mathbb{R}^n$ $(n \geq 2)$. Then $\mathbb{R}^n - A$ is connected.

Theorem 8.11 Let X be a countable, regular, T_1 space. Then X is not connected.

Theorem 8.12 Let X be a connected space, C a connected subset of X, and $X - C = A \mid B$. Then $A \cup C$ and $B \cup C$ are each connected.

Definition 8.2 Let X be a space and $p \in X$. The *component of p in X* is the union of all connected subsets of X which contain p.

Theorem 8.13 Each component of X is connected and closed.

Theorem 8.14 Let A and B be closed subsets of a compact, Hausdorff space X such that no component intersects both A and B. Then $X = H \mid K$ where $A \subset H$ and $B \subset K$.

Example 8.2 . This example will demonstrate the necessity of the "compactness" hypothesis of Theorem 8.14. Let X be the subset of $\mathbb{R}^2$ equal to $([0,1] \times \bigcup_{i\in\omega}\{1/i\}) \cup \{(0,0),(1,0)\}$. Show that the conclusion to Theorem 8.14 fails when $A = \{(0,0)\}$ and $B = \{(1,0)\}$.

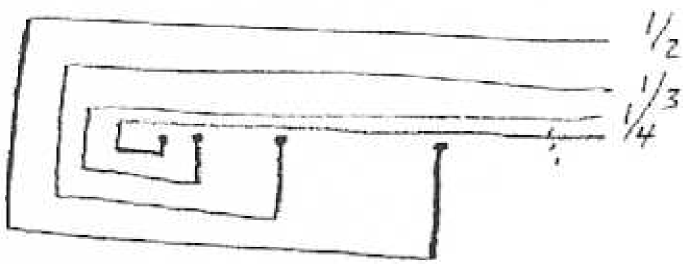
Definition 8.3 A *continuum* is a connected, compact, Hausdorff space.

Theorem 8.15 Let U be a proper, open subset of a continuum X. Then each component of $\overline{U}$ contains a point of $\mathrm{Bd}\,U$. (Note: $\mathrm{Bd}\,U = \overline{U} - U$.)

Theorem 8.16 ("To the boundary" theorem.) Let U be a proper, open subset of a continuum X. Then each component of U has a limit point on $\mathrm{Bd}\,U$.

Theorem 8.17 No continuum X is the union of a countable number (> 1) of disjoint closed subsets.

Example 8.3 This example shows the necessity of the compactness hypothesis on X.



The example X pictured above is a subset of the plane which is the union of a countable number of arcs as shown. Show that X is connected.

Theorem 8.18 Let $\{C_i\}_{i\in\omega}$ be a collection of continua such that for each i, $C_{i+1} \subset C_i$. Then $\bigcap_{i\in\omega} C_i$ is a continuum.

Theorem 8.19 Let $\{C_\alpha\}_{\alpha\in\lambda}$ be a collection of continua indexed by a well-ordered set λ such that if $\alpha < \beta$, then $C_\beta \subset C_\alpha$. Then $\bigcap_{\alpha\in\lambda} C_\alpha$ is a continuum.

Definition 8.4 Let X be a connected set. A point p in X is a *non-separating point* if and only if $X - \{p\}$ is connected. Otherwise p is a *separating* point.

Theorem 8.20 Let X be a continuum, p be a point of X, and $X - \{p\} = H \mid K$. Then $H \cup \{p\}$ is a continuum and if $q \neq p$ is a non-separating point of $H \cup \{p\}$, then q is a non-separating point of X.

Theorem 8.21 Let X be a metric continuum. Then X has at least two non-separating points.

Theorem 8.22 Let X be a continuum. Then X has at least two non-separating points.

Theorem 8.23 Let X be a metric continuum with exactly two non-separating points. Then X is homeomorphic to $[0, 1]$.

Definition 8.5 A space X is *locally connected at the point* p of X if and only if for each open set U containing p, there is a connected open set V such that $p \in V \subset U$. A space X is *locally connected* if and only if it is locally connected at each point.

Theorem 8.24 The following are equivalent:

X is locally connected.

X has a basis of connected open sets.

For each ρ in X and open set U containing ρ, the component of ρ in U is open.

For each ρ in X and open set U containing ρ, there is a connected set C so that $\rho \in \text{Int } C \subset C \subset U$.

For each ρ in X and open set U containing ρ, there is an open set V containing ρ and $V \subset$ (the component of ρ in U).

Theorem 8.25 Let X be a locally connected space and $f : X \to Y$ be an onto, closed or open map. Then Y is locally connected.

Definition 8.6 A *Peano Continuum* is a locally connected metric continuum.

Theorem 8.26 A Hausdorff space X is a Peano Continuum if and only if X is the image of $[0, 1]$ under a continuous function.

Definition 8.7

1. A space X is *arc-wise connected* if and only if for each pair of points $\rho, q \in X$ there is an embedding $h : [0, 1] \to X$ such that $h(0) = \rho$ and $h(1) = q$.
2. A space X is *locally arc-wise connected at* ρ if and only if for each open set U containing ρ there is an open set V containing ρ such that for each pair of points $x, y \in V$, there is an arc in U which contains x and y. (Note: "an arc" means the homeomorphic image of $[0, 1]$).
3. A space is *locally arc-wise connected* if and only if is locally arc-wise connected at each point.

Theorem 8.27 An arc-wise connected space is connected.

Theorem 8.28 A locally arc-wise connected space is locally connected.

Theorem 8.29 A Peano Continuum is arc-wise connected and locally arc-wise connected.

Theorem 8.30 An open, connected subset of a Peano Continuum is arc-wise connected.