Fundamental Group

Definition. Given topological spaces X, Y and $S \subset X$, then two continuous functions $f, g : X \to Y$ are *homotopic relative to* S iff there is a continuous function $H : X \times [0,1] \to Y$ such that

$$H(x,1) = g(x) \text{ for every } x \in X ;$$
$$H(x,0) = f(x) \text{ for every } x \in X ; \text{ and}$$
$$H(x,t) = f(x) = g(x) \text{ for every } x \in S \text{ and } t \in [0,1] .$$

Theorem FG.1. Given topological spaces X, Y and $S \subset X$, show that being homotopic relative to S is an equivalence relation on the set of all continuous functions from X to Y.

Definition. A continuous function $\alpha : [0,1] \to X$ is a *path*. If $\alpha(0) = \alpha(1) = x_0$, then α is a *loop* (or *closed path*) *based at* x_0.

Definition. Let α be a path, then α^{-1} is the path defined by $\alpha^{-1}(t) = \alpha(1-t)$. Two paths α, β are *equivalent*, denoted $\alpha \sim \beta$, iff α and β are homotopic relative to $\{0,1\}$. Denote the equivalence class of paths equivalent to α by $[\alpha]$.

Definition. Let α, β be paths with $\alpha(1) = \beta(0)$. Then their *product*, denoted $\alpha \cdot \beta$, is the path defined by

$$\alpha \cdot \beta(t) = \begin{cases} \alpha(2t) & , \quad 0 \le t \le \frac{1}{2} ; \\ \beta(2t-1) & , \quad \frac{1}{2} < t \le 1 . \end{cases}$$

Theorem FG.2. If $\alpha \sim \alpha'$, then $\beta \cdot \alpha \sim \beta \cdot \alpha'$. (Thus products of paths can be extended to products of equivalence relations.)

Theorem FG.3. Given α, β, and γ, then $(\alpha \cdot \beta) \cdot \gamma \sim \alpha \cdot (\beta \cdot \gamma)$ and $\alpha^{-1} \cdot \alpha \sim$ constant map.

Theorem FG.4. Let $x_0 \in X$, a topological space. Then the set of equivalence classes of loops based at x_0 with binary operation $[\alpha][\beta] = [\alpha \cdot \beta]$ is a group.

Definition. The above mentioned group is called the *Fundamental Group of X based at* x_0 and is denoted $\pi_1(X, x_0)$.

Theorem FG.5. Suppose X is a topological space and $p, q \in X$ lie in the same path component. Then $\pi_1(X, p)$ is isomorphic to $\pi_1(X, q)$.

Corollary FG.6. If X is path connected, then $\pi_1(X, p) \cong \pi_1(X, q)$ for any points $p, q \in X$.

Hence, for path connected spaces X, we sometimes just write $\pi_1(X)$ for the fundamental group.

Examples.

1. $\pi_1([0,1]) = 1$
2. $\pi_1(S^2) = 1$
3. $\pi_1(S^1) = \mathbb{Z}$
4. π_1 (cone over Hawaiian earring) $= 1$.

Definition. Let $f : X \to Y$ be a continuous function. Then $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ defined by $f_*([\alpha]) = [f \circ \alpha]$ is called the *induced homomorphism on fundamental groups.* (Check that f_* is well-defined.)

Theorem FG.7. If $g : X \to Y$, $f : Y \to Z$ are continuous functions, then $(f \circ g)_* = f_* \circ g_*$.

Theorem FG.8. If $f, g : X \to Y$ are continuous functions and $f \sim g$, then $f_* = g_*$.

Definition. Let $A \subset X$. Then $r : X \to A$ is a *strong deformation retract* iff there is a homotopy $R : X \times [0,1] \to X$ such that $R(x, 0) = x$ and $R(x, 1) = r(x)$, for all $x \in X$; and $R(a, t) = a$ for each $a \in A$ and $t \in [0,1]$.

Theorem FG.9. If $r : X \to A$ is a strong deformation retract and $a \in A$, then $\pi_1(X, a) \cong \pi_1(A, a)$.

Definitions.

1. X is *contractible* iff the identity map of X is homotopic to a constant map.
2. If X is path connected and $\pi_1(X) = 1$, then X is *simply connected*.

Theorem FG.10. A contractible space is simply connected.

Theorem FG.11. Let X, Y be path connected spaces. Then $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.

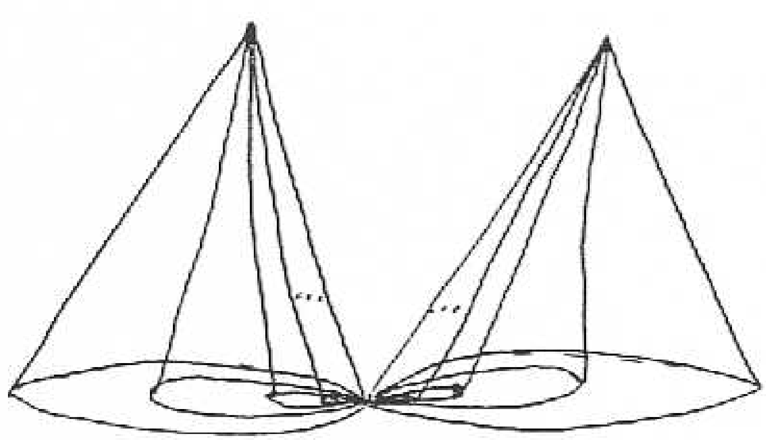
Theorem FG.12. Let $X = U \cup V$ where U and V are open, path connected, and simply connected subsets of X and $U \cap V$ is path connected. Then X is simply connected.

Theorem FG.13. Let $X = U \cup V$, where U, V are open, path connected subsets of X, $U \cap V$ is path connected and simply connected, and $x \in U \cap V$. Then $\pi_1(X, x) \cong \pi_1(U, x) * \pi_1(V, x)$.

Theorem FG.14. π_1 (projective plane) $\cong \mathbb{Z}_2$.

Theorem FG.15. $\pi_1(\infty) \cong \mathbb{Z} * \mathbb{Z}$.

Example. Let X be two cones over the Hawaiian earring identified at a point as in the figure below. Show that $\pi(X) \ncong 1$.



Theorem FG.16. (Van Kampen's Theorem). Let $X = U \cup V$, where U, V are open and path connected and $U \cap V$ is path connected and non-empty. Let $x \in U \cap V$. Then $\pi_1(X,x) \cong \frac{\pi_1(U,x) * \pi_1(V,x)}{N}$ where N is the smallest normal subgroup containing $\{i_*(\alpha) j_*(\alpha^{-1})\}_{\alpha \in \pi_1(U \cap V, x)}$ and i, j are the inclusion maps of $U \cap V$ in U and V respectively.

Exercise. Compute π_1 (Klein Bottle).