Stellar Stability and the Chandrashekar Limit

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March 23, 2006

Abstract

One main application of the theory of general relativity is its application to the stability criteria of stellar structures and their consequences. We discuss a popular relativistic model of spherically symmetric stars, and, in the Newtonian limit, we derive the famous Chandrashekar limit, the upper bound for the mass of white dwarves. The Chandrashekar limit is derived to be 1.26 times the mass of the sun[1].

1 Introduction

The success of the Newtonian theory of gravity comes with its ability to describe the planetary orbits and gravitational phenomena around earth. However, with Einstein's general relativity and its success in describing phenomena such as the bending of light around massive objects, perihelion of Mercury, and many other general relativistic phenomena[2], one comes to realize that Newtonian gravity is only correct in the limit of weak, linearized Einstein's equation.

Naturally, this leads one to wonder whether general relativity is able to tell us more about stability criteria of stellar structures. This issue, with its Newtonian correspondence, is taken up in sections 2 and 3.

In the second half of this document we will focus our attention on the discussion of Newtonian stars by taking certain limits of results derived in sections 2 and 3 , as most of the stars in the sky are adequately described by Newtonian gravity. We look at the stability of an idealized class of Newtonian stars: the polytropes. This is discussed in section 4.

There is a very common class of stars that reduce to polytropes in certain limits, and to which it is believed that the end life of our sun belongs.

They are known as the white dwarves. A white dwarf is an astronomical object which is produced when a low to medium mass star dies. These stars are not heavy enough to generate the core temperatures required to fuse carbon in nucleosynthesis reactions. After one has become a red giant during its helium-burning phase, it will shed its outer layers to form a planetary nebula, leaving behind an inert core consisting mostly of carbon and oxygen.[3] This core has no further source of energy, and so will gradually radiate away its energy and cool down to extremely low temperatures. The core, no longer supported against gravitational collapse by fusion reactions, becomes extremely dense, with a typical mass of that of the sun contained in a volume about equal to that of the Earth. This core is only supported against gravitational collapse by the degeneracy pressure, a phenomenon resulted from the quantum mechanical Pauli's exclusion principle. However, if the mass of the white dwarf is too heavy, then the gravitational is too great for the degeneracy pressure, then the white dwarf would inevitably collapse. Therefore, there exist a limit for the mass of the white dwarf above which it would no longer be stable. This limit is known as the Chandrasekhar limit. We present a particular derivation of this limit in section 5.

2 The Stellar Model and the Governing Differential Equations

In this section we examine a model where the star is spherically symmetric, static, and isentropic.

2.1 The Metric and the Einstein Equation

In this section we seek to set up the general model that we will be using throughout this document. We assume the star to be spherically symmetric and static. In particular, we assume metric of the following form:

$$g_{tt} = -B(r), g_{rr} = A(r), g_{\theta\theta} = r^2, g_{\phi\phi} = r^2 sin^2(\theta). \tag{1}$$

All other components of the metric vanish. The energy-momentum tensor is assumed to of a perfect fluid:

$$T_{\mu\nu} = pg_{\mu\nu} + (p + \rho)U_\mu U_\nu. \tag{2}$$

,where p denotes the proper pressure, ρ the proper energy density, and U^μ the velocity four-vector. Since the fluid is at rest, we can take

$$U_r = U_\theta = U_\phi = 0; U_t = -\sqrt{B(r)}. \tag{3}$$

Note that the assumption of time independence and spherical symmetry implies that p and ρ are functions of r only. Now, with this metric in hand, we are able to solve for the Ricci tensor and the Einstein equation:

$$R_{tt} = -\frac{B''}{2A} + \frac{B'}{4A}(\frac{A'}{A} + \frac{B'}{B}) - \frac{B'}{rA} = -4\pi G(\rho + 3p)B \tag{4}$$

$$R_{rr} = \frac{B''}{2B} - \frac{B'}{4B}(\frac{A'}{A} + \frac{B'}{B}) - \frac{A'}{rA} = -4\pi G(\rho - p)A \tag{5}$$

$$R_{\theta\theta} = -1 + \frac{r}{2A}(-\frac{A'}{A} + \frac{B'}{B}) + \frac{1}{A} = -4\pi G(\rho - p)r^2 \tag{6}$$

The $R_{\phi\phi}$ equation is the same as the $R_{\theta\theta}$ equation. a prime denotes d/dr. Now, hydrostatic equilibrium would imply the condition[1]:

$$\frac{B'}{B} = -\frac{2p'}{p+\rho} \tag{7}$$

We will employ equations 4, 5, 6, and 7 to derive a governing differential equation of state in the next section.

2.2 The Differential Equation

The first step to derive a governing differential equation is to derive an equation for A(r) alone:

$$\frac{R_{rr}}{2A} + \frac{R_{\theta\theta}}{r^2} + \frac{R_{tt}}{2B} = -\frac{A'}{rA^2} - \frac{1}{r^2} + \frac{1}{Ar^2} = -8\pi G\rho \tag{8}$$

Equation 8 has a solution:

$$A(r) = [1 - \frac{2GM(r)}{r}]^{-1} \tag{9}$$

where

$$M(r) = \int_0^r 4\pi r'^2 \rho(r') dr' \tag{10}$$

Now, we can use equations 7 and 9 to eliminate A(r) and B(r) from equation 6. Rearranging the result gives us the following differential equation:

$$-r^2 p^{'}(r) = GM(r)\rho(r)[1 + \frac{p(r)}{\rho(r)}][1 + \frac{4\pi r^3 p(r)}{M(r)}][1 - \frac{2GM(r)}{r}]^{-1} \tag{11}$$

Note that this differential equation is essentially the Newtonian gravitational differential equation for stellar structure, with the last 3 factors as the general relativistic corrections.

Our work is not done. We now assume the star to be isentropic. i.e. the entropy per nucleon s does not vary throughout the star. This assumption holds for two different types of stars: (1) Stars at absolute zero. In this case, the entropy per nucleon will be zero throughout the star, and (2) stars in perfect convective equilibrium. This assumption is important because the pressure p in general is a function of the density ρ, entropy per nucleon s. Therefore, with this assumption, p(r) may be regarded as a function of $\rho(r)$, alone, with no explicit dependence on r. Now if we rewrite equation 10 as:

$$M^{'}(r) = 4\pi r^2 \rho(r); M(0) = 0 \tag{12}$$

Equations 11 and 12, together with an equation of state $p(\rho)$ and initial value $rho(0)$, we can determine $rho(r)$, M(r), and p(r) throughout the star.

2.3 Useful Quantities

The main results of the model are equations 11 and 12, the condition for which the star is at an equilibrium. Here based on this model, we define a few useful quantities that will make the future discussion easier.

First we need a definition of the total nucleon number N:

$$N \equiv \int \sqrt{g} J_N^0 dr d\theta d\phi \tag{13}$$

where J_N^μ is the nucleon number current. We can in fact re-express N as the following:

$$N = \int_0^R 4\pi r^2 \sqrt{A(r)} n(r) dr d\theta d\phi = \int_0^R 4\pi r^2 [1 - \frac{2GM(r)}{r}]^{-1/2} n(r) dr d\theta d\phi \tag{14}$$

where R is the radius of the star, and n(r) is the proper nucleon number density.

Then we seek a useful definition of the internal energy of the star. Intuition guides us that the total energy M(r) is a sum of energy of matter and the internal energy of the star. Then, if we define the internal energy E as the following:

$$E \equiv M - m_N N \tag{15}$$

where m_N is the rest mass of a nucleon, and N is the number of nucleons in the star. Therefore, the second term in equation **??** can be interpreted as the energy of matter, and therefore E can be interpreted as the internal energy of the star. Similarly, we can also define the proper internal energy density as:

$$e \equiv \rho(r) - m_N n(r) \tag{16}$$

3 Stellar Stability

Solutions to equations 11 and 12 are equilibrium states of the star. However, the equilibrium states can either be stable or unstable. For most purposes, only the stable solutions are of interest in applications. Therefore, we need conditions that can tell us whether an equilibrium state is stable.

In order to tell if a particular equilibrium is stable, the general procedure is to perform linear perturbation around the equilibrium point, and compute the frequencies ω_n of the normal modes. If the frequency has a positive imaginary part, then the time variation factor $exp(-i\omega_n t)$ would be exponentially growing, and the equilibrium would be unstable. However, this general test is often tedious. Here we quote[1, 4] two useful theorems without proof.

Theorem 1. A star consisting of a perfect fluid with constant chemical composition and entropy per nucleon, can only pass from stability to instability with respect to some particular radial normal mode, at a value of the central density $\rho(0)$ for which the equilibrium energy E and nucleon number N are stationary, that is,

$$\frac{\partial E(\rho(0); s, ...)}{\partial \rho(0)}; \frac{\partial N(\rho(0); s, ...)}{\partial \rho(0)} \tag{17}$$

By a "radial" normal mode is meant a mode of oscillation in which the density perturbation $\delta\rho$ is a function of r and t alone.

The intuition is that ω_n^2 is a real continuous function of $\rho(0)$. If $\omega_n^2 > 0$, then there are two stable modes. However, if $\omega_n^2 < 0$, then one of the two modes is unstable. Therefore, the transition can only occur at a value of $\rho(0)$ for which ω_n^2 vanishes, and at this point it gives the above condition.

The second theorem is a variational principle:

Theorem 2. A particular stellar configuration, with uniform entropy per nucleon and chemical composition, will satisfy equations 11 and 12 for equilibrium, if and only if M(r) is stationary with respect to all variations of $\rho(r)$ that leave N, and entropy per nucleon unchanged. The equilibrium is stable with respect to radial oscillations if and only if M, or equivalently E, is a minimum with respect to all such variations.

The intuition is simply that, if the equilibrium is at a local minimum of energy with respect to small perturbations, then the system is stable.

We will employ these theorems as we discuss the stability of Newtonian stars.

4 Newtonian Stars and the Polytropes

To examine the behavior of Newtonian stars we first note that at the Newtonian limit, equation 11 reduces to:

$$-r^2 p^{'}(r) = GM(r)\rho(r). \tag{18}$$

Combining with equation 12, we can obtain:

$$\frac{d}{dr}\frac{r^2}{\rho(r)}\frac{dp(r)}{dr} = -4\pi G r^2 \rho(r) \tag{19}$$

This is the governing differential equation for all spherically symmetric, static Newtonian stars. Note that in this case M(r) becomes the mass of the star. As commented in section 2, we need an initial value $\rho(0)$ as well as an equation of state $p(\rho(r))$ to solve the full dynamics. We now provide such an equation of state. For many stars, the internal energy density is proportional to the pressure, so:

$$e \equiv \rho - m_N n = (\gamma - 1)^{-1} p \tag{20}$$

where γ is some constant. In the case of that both e and p are proportional to the temperature, then γ is in fact the ratio of specific heats. If now impose the condition of uniform entropy per nucleon, we can derive the following equation of state:

$$p = K\rho^r \tag{21}$$

where K is some constant. In general K depends on the entropy per nucleon and the chemical composition. Any star that has equation 21 as the equation of state is called a *polytrope*. Now given $\rho(0)$, we are ready to solve all physical properties of the star. In particular, we are interested in the mass of the star M, radius of the star R, and the total internal energy E. Applying equations 19 and 21 we obtain:

$$M = 4\pi\rho(0)^{(3\gamma-4)/2}(\frac{K\gamma}{4\pi G(\gamma-1)})^{3/2}\xi_1^2|\theta'(\xi_1)| \tag{22}$$

$$R = (\frac{K\gamma}{4\pi G(\gamma-1)})^{1/2}\rho(0)^{(\gamma-2)/2}\xi_1 \tag{23}$$

where $\theta(\xi)$ is the **Lane-Emden function** [5] of index $(\gamma-1)^{-1}$. For $\gamma > 6/5$, $\theta(\xi)$ has a zero, denoted by ξ_1. The expression for the internal energy is rather simple, given as:

$$E = -\frac{3(\gamma-1)}{5\gamma-6}\frac{GM^2}{R} \tag{24}$$

With the expression for internal energy in hand, we can examine the stability condition for polytropes. If we apply theorem 2 from section 3, and then if we define:

$$\rho_{crit} = (\frac{M^{2/3}G(4\pi/3)^{1/3}}{5K})^{1/(\gamma-4/3)} \tag{25}$$

then for $\gamma > 4/3$, E has a minimum at ρ_{crit}, corresponding to stable equilibria. For $\gamma > 4/3$, E has a maximum at ρ_{crit}, corresponding to unstable equilibria.

We will apply the results obtained in this section to derive the Chandrasekhar Limit in section 5.

5 White Dwarves and the Chandrasekhar Limit

As mentioned in section 1, a white dwarf is an aged star that has exhausted its nuclear fuel and begins to cool and contract. As energy will eventually radiate away, and the star has lost its ability to generate energy from within, the temperature will be extremely low. This means that the electrons will all be at the lowest possible states. However, according to quantum mechanics, specifically the Pauli exclusion principle, no two electrons can occupy the same quantum state. This implies that there are at most 2 electrons per energy level(for two different possible spins). If we model the electrons of the star as particles in a box, we have $4\pi k^2(2\pi\bar{h})^{-3}dk$ levels of energy per unit volume with momenta between k and k + dk. Therefore, we can derive the expressions for internal energy density e and pressure p :

$$e = (\frac{8\pi}{(2\pi\bar{h})^3}) \int_0^{k_f} [(k^2 + m_e^2)^{1/2} - m_e]k^2 dk \tag{26}$$

$$p = (\frac{8\pi}{(2\pi\bar{h})^3}) \int_0^{k_f} \frac{k^4}{(k^2 + m_e^2)^{1/2}} dk \tag{27}$$

The equation of state relating e and p is not simple in this case. However, it does reduce to that of a polytrope : (A) $\rho \ll \rho_c$ and (B) $\rho \gg \rho_c$, where ρ_c is the critical density at which k_f becomes equal to m_e.

Case (A) in the limit of $\rho \ll \rho_c$, we get a polytrope with

$$\gamma = \frac{5}{3}; K = \frac{\bar{h}^2}{15 m_e \pi^2} (\frac{3\pi^2 \rho}{m_N \mu})^{5/3} \tag{28}$$

where μ is the number of nucleons per electron. $\mu \simeq 2$ for stars that have used up their hydrogen. Since $\gamma > 4/3$, we conclude that the star is stable in the limit of low density, as one might intuitively expect.

Case (B) in the limit of $\rho \gg \rho_c$, we get a polytrope with

$$\gamma = \frac{4}{3}; K = \frac{\bar{h}}{12\pi^2} (\frac{3\pi^2}{m_N \mu})^{4/3} \tag{29}$$

Note that for this case, $\gamma = 4/3$, so this is in fact at the transition between stability and instability. From equation 22 we can calculate the mass for this polytrope. This comes out to be:

$$M = 5.87\mu^{-2} M_{\odot} \tag{30}$$

where $M_\odot$ denotes the mass of the sun.

Therefore, we see that, in the limit of low density, the star is stable. As the mass of the star grows monotonically with the increase of density. in the limit of $\rho \to \infty$, the mass of the star approaches $M = 5.87\mu^{-2}M_\odot$. Therefore, the white dwarf exists and is stable for $M \leq 5.87\mu^{-2}M_\odot$. This maximum limit is known as the **Chandrasekhar limit**. For $\mu = \frac{56}{26}$, this limit is given as $1.26M_\odot$. Further investigation shows that if a dying star exceeds this mass limit, then instead of becoming a white dwarf, it would either become a neutron star or a blackhole.

References

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