1 Introduction

General relativity predicts, among other things, that waves in the metric of spacetime form a valid solution to the equations governing gravitation. Since this prediction was made, much effort has been put into various attempts to detect this gravitational radiation, or gravitational waves. There are two main reasons one might wish to detect gravitational radiation. Firstly, attempts to detect gravitational waves serve as a test of general relativity. Gravitational waves are a somewhat surprising prediction of relativity which is distinctly non-Newtonian (or post-Newtonian, as the literature would have it.) If gravitational waves are detected, their existence would provide a partial confirmation of the theory. Alternatively, if gravitational waves were not detected in systems where they are expected, this result would suggest that the theory is incomplete or wrong. Secondly, if gravitational radiation could be reliably detected, then it would provide a new method with which to observe the universe. Astronomers speculate that the study of gravitational radiation from such objects as supernovae and quasars would give a much better understanding of those systems. There is also the possibility that gravitational radiation will allow us to discover and study completely new phenomena.

In this report, I will focus primarily on the successful detection of gravitational radiation in the PSR1913+16 binary pulsar system discovered by Hulse and Taylor in 1974, and show that this result confirmed the predictions of general relativity for a binary system. Section 2 provides a brief review of gravitational waves; section 3 discusses the expected effects and sources of gravitational radiation, and reviews some techniques currently being employed to attempt the direct detection of gravitational radiation. Finally, section 4 discusses the indirect detection of gravitational radiation in the binary pulsar system.

2 Background

The material in this section is derived largely from Schutz [5], but also from Wald [11]. Einstein's equation states that the gravitational tensor $G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}$ is governed by

$$G_{ab} = 8\pi T_{ab} \tag{1}$$

where T_{ab} is the stress energy tensor. In the case where gravity is comparably weak, spacetime is nearly flat, so we can choose coordinates in which the metric g_{ab} is close to the metric η_{ab} of flat spacetime. Therefore we have

$$g_{ab} = \eta_{ab} + h_{ab} \tag{2}$$

where h_{ab} is a very small correction to the flat spacetime metric. A change of coordinates

$$x^a \to x^a + \xi^a(x^b) \tag{3}$$

leaves these equations unchanged, and can be expressed equivalently as a change to h_{ab}:

$$h_{ab} \to h_{ab} - \xi_{a,b} - \xi_{b,a} \tag{4}$$

This gauge transformation can be used to simplify Einstein's equation by choosing ξ such that

$$\overline{h}^{\mu\nu}_{,\nu} = 0 \tag{5}$$

where $\overline{h}^{\alpha\beta} = h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}h$, and h is the trace $h = h^{\alpha}_{\alpha} = \eta^{\alpha\beta}h_{\beta\alpha}$ of h_{ab}. In this gauge, $G^{ab} = -\frac{1}{2}\triangle\overline{h}^{\mu\nu}$ to first order, where $\triangle$ is the D'Alembertian operator, defined by

$$\triangle f = (-\frac{\partial^2}{\partial t^2} + \nabla^2)f = \eta^{\mu\nu}f_{,mn} \tag{6}$$

The D'Alembertian operator is also sometimes represented by a square, but I can't figure out how to make a square in LaTeX, so here we are. Returning to the topic at hand, Einstein's equation then gives

$$\triangle\overline{h}^{\mu\nu} = -16\pi T^{\mu\nu} \tag{7}$$

This is just an inhomogeneous three dimensional wave equation in each component of $\overline{h}^{\mu\nu}$. In a vacuum, $T^{\mu\nu} = 0$, so we expect solutions of the form

$$\overline{h}^{\alpha\beta} = A^{\alpha\beta}e^{ik_\mu x^\mu} \tag{8}$$

These oscillations in the metric are called gravitational waves. Rewriting equation (7) using equation (6), and plugging in equation (8), we can obtain the condition that k^ν be a null vector. Furthermore, equation (5) can be used to give the condition $A^{\alpha\beta}k_\beta = 0$, so the amplitudes $A^{\alpha\beta}$ must be orthogonal to the direction of travel k. Additionally, further gauge transformations of the form (4), with $(-\frac{\partial^2}{\partial t^2} + \nabla^2)\xi_\alpha = 0$, can be used to impose conditions

$$A^{\alpha}_{\alpha} = 0 \text{ and } A_{\alpha\beta}U^\beta = 0 \tag{9}$$

for some fixed four-velocity U. Choosing U to be the time basis vector, and letting the direction of wave travel be in the z-direction, we can express A in matrix form. Then only $A_{yx} = A_{xy}$, and $A_{xx} = A_{yy}$ are nonzero. This last set of gauge transformations, together with the previous gauges, is called a transverse traceless set of gauge conditions, and the radiation gauges are usually denoted $h^{TT}_{\alpha\beta}$. However, for the rest of this report, I will be using the transverse traceless gauge unless otherwise indicated, so I will simply use $h_{\alpha\beta}$ to denote the transverse traceless gauge.

3 Effects and Sources

Gravitational waves are clearly a phenomenon not predicted by the Newtonian theory of gravity, but it remains to check precisely what effect a gravitational wave might have on the motion of massive particles. To do this, consider two free particles separated by a vector ξ, orthogonal to the direction of wave travel. Then ξ obeys the geodesic deviation equation

$$\frac{d^2}{d\tau^2}\xi^a = R^{\alpha}_{\mu\nu\beta}U^{\mu}U^{\nu}\xi^{\beta} \tag{10}$$

where U is the four-velocity of the two particles [5]. To first order, we have $U \approx (1, 0, 0, 0)$, so the above equation simplifies to

$$\frac{\partial^2}{\partial t^2}\xi^a = R^{\alpha}_{00\beta}\xi^{\beta} \tag{11}$$

Plugging equation (2) into the expression for the Riemann curvature tensor gives the following expression for $R_{\alpha\beta\mu\nu}$

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}(h_{\alpha\nu,\beta\mu} + h_{\beta\mu,\alpha\nu} - h_{\alpha\mu,\beta\nu} - h_{\beta\nu,\alpha\mu}) \tag{12}$$

In the transverse traceless gauge, we can then calculate the relevant components of the Riemann tensor to be

$$R^{\alpha}_{00\beta} = \frac{1}{2}\frac{\partial^2 h_{\alpha\beta}}{\partial t^2} \tag{13}$$

so

$$\frac{\partial^2}{\partial t^2}\xi^a = \frac{1}{2}\frac{\partial^2 h_{\alpha\beta}}{\partial t^2}\xi^{\beta} \tag{14}$$

Choosing coordinates such that ξ is in the x direction, and approximating ξ^{α} by $(0, \epsilon, 0, 0)$ gives

$$\frac{\partial^2}{\partial t^2}\xi^x = \frac{1}{2}\epsilon\frac{\partial^2 h_{xx}}{\partial t^2} \tag{15}$$

and

$$\frac{\partial^2}{\partial t^2}\xi^y = \frac{1}{2}\epsilon\frac{\partial^2 h_{xy}}{\partial t^2} \tag{16}$$

where the wave travels in the z direction [5].

Since $h_{\alpha\beta}$ oscillate with time, we should expect the particles to behave as if experiencing a small periodic force. The amplitude of this force depends on the degree of separation ϵ, but also more importantly on the radiation gauge components $h_{\alpha\beta}$. The expected size of $h_{\alpha\beta}$ depend in turn on the source of the waves. In general, nearby astronomical events are expected to generate gravitational waves far too small to be detected using current technology. Given two identical masses of mass m, at an initial distance L_0 apart, oscillating with an amplitude A and frequency ω, a calculation of Schutz shows that the expected value of $\overline{h}_{xx}$ is given by

$$\overline{h}_{xx} = \frac{-2m\omega^2}{r}(L_0 A\cos(\omega(r-t)) + 2A^2 cos(2\omega(r-t))) \tag{17}$$

where r is the distance of the observer from the centre of this apparatus [5]. However, the above equation can be used to get some good order of magnitude estimates for gravitational wave effects from various sources. Schutz uses the values $m = 10^3\text{kg} = 7\times10^{-24}m$, $L_0 = 1m$, $A = 10^{-4}m$, and $\omega = 10^4 s^{-1} = 3\times10^{-4}m$ to produce the result that $\overline{h}_{xx}$ has oscillations of amplitude $a \simeq \frac{10^{-34}}{r}$ [5]. The energy carried by this wave, in Joules, is given by $U = \frac{1}{2}c^2\omega^2 a^2$ where a is the amplitude of oscillations in $\overline{h}_{xx}$ [3]. Thus for the wave with amplitude $\frac{10^{-34}}{r}$, and frequency $\omega = 10^4 s^{-1}$ we get energy on the order of $10^{-44}J$, which are far too small to be detected. Generally speaking, then, events on or around the Earth do not produce detectable gravitational radiation. Scaling up the masses, distances, and speeds to astronomical scales, involves increasing the magnitudes of m, L_0, and A, but also decreasing ω and increasing r, so a system of orbiting stars, approximated by a set of sun-like masses on springs, might have effects on the order of $10^{-38}J$ which are still miniscule. Of course, a system of orbiting stars (or whatever) looks very little like a set of masses on springs, and so this calculation is obviously flawed, but it gives an idea of the weakness of gravitational wave effects as expected on Earth.

More serious calculations of the gravitational radiation expected to arrive at Earth from various astronomical phenomena are generally based on estimates about poorly understood astrophysical processes, and thus are also flawed, though perhaps not as badly as the one above. This is generally readily admitted by those who make such attempts; Thorne [9] points out that prior to X-ray astronomy by instruments outside the atmosphere, estimates of X-ray radiation in space proved to be very inaccurate. Nevertheless, it is generally expected that gravitational radiation in the neighbourhood of Earth will not have effects on an order of greater than 10^{-16} on $h_{\alpha\beta}$, and thus will be very difficult to detect [9]. Examples of events violent enough to create gravitational radiation that might be detectable here are black hole and neutron star births in the Milky Way (effects on the order of $10^{-19} - 10^{-17}$ at Earth), supermassive black hole collisions ($10^{-19} - 10^{-16}$), globular cluster black hole collisions ($10^{-21} - 10^{-19}$), and neutron star corequakes ($10^{-24} - 10^{-20}$) [9].

Various methods based on laser interferometry have been used to attempt to detect gravitational radiation directly, but the scales involved are so miniscule as to push the limits of current technology. At the levels of precision needed, the detectors are also far more likely to pick up various kinds of noise, including seismic and thermal noise. Thermal noise can be reduced by supercooling; seismic noise is more problematic [2], [11]. Spaceborne detectors overcome this problem, but fall victim to new sources of noise, including solar radiation pressure [2]. One method employed in an attempt to increase the detectability of such small oscillations is to set up something like the mass-spring the system described above, but place the masses initially at rest, and look for vibrations due to incoming gravitational waves [5], [11]. If the frequency of the incoming gravitational waves is near the resonant frequency of the system, oscillations can be amplified to create detectable events. However, there are several practical problems with this approach. Firstly, we need to estimate the frequency of incoming gravitational waves, which is again based on poorly understood astrophysical processes [9]. Secondly, noise of the right frequency is also amplified, again causing problems. A third method of gravitational wave detection involves Doppler tracking of spacecraft. Here again, however, various sources of noise reduce the effectiveness of gravitational wave detection [2].

Legions of experiments of this type have failed to conclusively detect gravitational waves in the vicinity of this planet. However, gravitational waves have been detected indirectly by a completely different kind of investigation altogether.

4 Binary Pulsars

Pulsars are spinning neutron stars, first discovered in the 1960's, which emit pulses of radiation at nearly constant intervals. In the mid 1970's, Russell Hulse and Joseph Taylor began a systematic survey of the sky from the Arecibo Observatory in Puerto Rico, looking for new pulsars. In 1974, they discovered "'an unusual pulsar"', designated PSR1913+16, which had a very short pulsation period (averaging 59 milliseconds between pulses) and experienced periodic variations of up to 80 microseconds per day in the pulsation rate, which is roughly 3000 times greater than any variation in a previously observed pulsar. Further study of the pulsar indicated that the periodic changes in the pulsation rate could be accounted for as the Doppler shifts expected if the pulsar were in orbit around a companion star [1]. From observations of these Doppler shifts over a number of years, Hulse and Taylor could calculate the orbital elements of the pulsar, including the orbital period P_b, rate of change of orbital period $\dot{P}_b$, sine of inclination angle $\sin i$, and eccentricity e. Hulse and Taylor immediately recognized the potential of the PSR1913+16 system as a test case for general relativity – it consists of "an accurate clock in a high speed, eccentric orbit in a strong gravitational field" [1].

In particular, the system of orbiting stars was expected to give off gravitational radiation, causing it to lose energy. This in turn would cause the stars to fall closer together and speed up, resulting in a smaller orbital period. The predictions of how much energy would be lost in a system of orbiting point masses had already been calculated by Peters and Mathews a decade earlier [3], [4].

Returning to the Einstein equation $G^{\alpha\beta} = -16\pi T^{\mu\nu}$, we can take $g_{ab} = \eta_{ab} + h_{ab}$, with the gauge conditions on h_{ab} specified earlier, and write out the Einstein equation in unlinearized form as

$$\triangle \overline{h}_{\alpha\beta} = -16\pi S_{\alpha\beta} \tag{18}$$

Here

$$S_{\alpha\beta} = T_{\alpha\beta} + \sum_{k=2}^{\infty} X_{\alpha\beta}^{(k)} \tag{19}$$

where $X_{\alpha\beta}^{(k)}$ are the sum of k^{th} degree terms in $h_{\alpha\beta}$. By the gauge conditions, we have $\overline{h}_{\alpha\beta,\alpha}$ vanishing, so $S_{\alpha\beta,\alpha} = 0$. This means that we can use the divergence equation to obtain conservation laws for components of $S_{\alpha\beta}$ [4]. In particular, integrating over some domain containing the system of interest, $\frac{d}{dt}\int S_{00}dV = \int S_{0i}dS_i$. The integral on the left is the total energy of the system, so we have

$$\frac{dE}{dt} = \int S_{0i}dS_i \tag{20}$$

Taking a large sphere completely enclosing the system, we can approximate $T_{0i} = 0$, since these terms should be negligible far from the masses. Therefore the only contributions to

S_{0i} come from $X_{0i}^{(k)}$. Now the exact solution to equation (18) is

$$\overline{h}_{\alpha\beta} = -4 \int \left[\frac{S_{\alpha\beta}(r', t)}{|r - r'|} \right]_{t-|r-r'|} dV' \tag{21}$$

so we expect $\overline{h}_{\alpha\beta}$ to be proportional to $\frac{1}{r}$ for large r [4]. Therefore we can discard all $X_{0i}^{(k)}$ except when $k = 2$ to obtain

$$\frac{dE}{dt} = \int X_{0i}^{(2)} dS_i \tag{22}$$

$\int X_{0i}^{(2)}$ can actually be written out explicitly in an incredibly ugly expression involving lots of terms [4]. However, this can be simplified considerably by recalling that the system under investigation is a periodic one. Assuming that the loss of energy is small, we can consider the average energy loss over one complete period of motion, and assume that the change in parameters of the system is negligible over that period [4]. Then any terms in $X_{0i}^{(2)}$ which are pure time derivatives can be discounted as negligible, and we obtain the expression

$$\int \frac{dE}{dt} dt = \frac{1}{32\pi} \int \int \sum_{\alpha,\beta} h_{\alpha\beta,0} \overline{h}_{\alpha\beta,i} dS_i dt \tag{23}$$

Converting the right side back into a volume integral, via the divergence theorem, yields

$$\int \frac{dE}{dt} dt = \frac{1}{32\pi} \int \int \sum_{\alpha,\beta} h_{\alpha\beta,0} \overline{h}_{\alpha\beta,ii} dV dt \tag{24}$$

Note that

$$\overline{h}_{\alpha\beta,ii} = \triangle \overline{h}_{\alpha\beta} = -16\pi S_{\alpha\beta} \tag{25}$$

so

$$\int \frac{dE}{dt} dt = -\frac{1}{2} \int \int \sum_{\alpha,\beta} h_{\alpha\beta,0} S_{\alpha\beta} dV dt \tag{26}$$

Assuming the velocities are small compared to the speed of light, we can express this in terms of the mass tensor

$$Q_{ij} = \sum_a m^a x_a^i x_a^i \tag{27}$$

where m^a are the masses in the system, with positions x_a [4]. We do this by taking equation (21) and expanding it in a Taylor series about the present time t, neglecting higher order terms. Then using the identity $\int S_{ij} dV = \frac{1}{2} \frac{d^2 Q_{ij}}{dt^2}$ gives us

$$\int \frac{dE}{dt} dt = -\frac{1}{5} \int \left[\frac{d^3 Q_{ij}}{dt^3} \frac{d^3 Q_{ij}}{dt^3} - \frac{d^3 Q_{ij}}{dt^3} \frac{d^3 Q_{jj}}{dt^3} \right] dt \tag{28}$$

The above approach is the one given in Peters [4]. This expression can also be derived using multipole expansion [3]. To calculate Q_{ij} for the binary system involves choosing coordinates for the two stars. Taking the centre of mass to be the origin, we can take coordinates $(d_1 \cos\psi, d_1 \sin\psi)$ and $(-d_2 \cos\psi, -d_2 \sin\psi)$ for masses m_1 and m_2 respectively. (Recall that all motion in a 2-body gravitational system takes place on a plane.) Then the d_i are related by $d = \frac{d_i(m_1+m_2)}{m_i}$, and

$$Q_{xx} = \mu d^2 \cos^2\psi \tag{29}$$

$$Q_{yy} = \mu d^2 \sin^2\psi \tag{30}$$

$$Q_{xy} = Q_{xy} = \mu d^2 \cos\psi \sin\psi \tag{31}$$

where μ is the reduced mass $\frac{m_1 m_2}{m_1+m_2}$. Assuming Newtonian motion, we have

$$d = \frac{a(1-e^2)}{1+e\cos\psi} \tag{32}$$

and

$$\psi = \frac{1}{d^2}\sqrt{(a(m_1+m_2)(1-e^2))} \tag{33}$$

Here e is the eccentricity of the orbit, and a is the length of the semimajor axis of the orbital ellipse. These equations allow us to calculate the appropriate time derivatives of Q_{ij} [3]:

$$\frac{d^3 Q_{xx}}{dt^3} = \beta(1+e\cos\psi)^2(2\sin 2\psi + 3e\sin\psi\cos^2\psi) \tag{34}$$

$$\frac{d^3 Q_{xx}}{dt^3} = -\beta(1+e\cos\psi)^2(2\sin 2\psi + e\sin\psi(1+3\cos^2\psi)) \tag{35}$$

and

$$\frac{d^3 Q_{xx}}{dt^3} = -\beta(1+e\cos\psi)^2(2\cos 2\psi - e\sin\psi(1-3\cos^2\psi)) \tag{36}$$

where

$$\beta^2 = \frac{4m_1^2 m_2^2(m_1+m_2)}{a^5(1-e^2)^5} \tag{37}$$

Plugging these into equation (28) and integrating, we get the rather improbable expression for time-averaged energy output over one period:

$$\overline{\frac{dE}{dt}} = -\frac{32}{5}\frac{m_1^2 m_2^2(m_1+m_2)}{a^5(1-e^2)^{\frac{7}{2}}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right) \tag{38}$$

If units are to be used where G and c are not 1, an extra factor of $\frac{G^4}{c^5}$ is needed in this expression [3].

How does this show up in the orbital motion of the stars? Wagoner used Peters' equations to derive the formula for rate of change of the period from Kepler's law $P = 2\pi a^{\frac{3}{2}}(G(m_1+m_2))^{\frac{-1}{2}}$. Here P is the period. Then the prediction of general relativity is given by the following expression [10].

$$\frac{1}{P}\frac{dP}{dt} = -\frac{96}{5}\frac{G^3 m_1 m_2 (m_1+m_2)}{a^4(1-e^2)^{\frac{7}{2}}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right) \tag{39}$$

At the time Wagoner was writing, the only orbital parameter in this equation which had been reliably measured was the eccentricity e. In particular, the masses of the two stars were undetermined [6]. However, by careful observations over several years, Taylor and his colleagues managed to deduce the value of a, and place constraints on the values of m_1 and m_2 sufficient to calculate the expected value of $\frac{dP}{dt}$ as $(-2.404 \pm 0.003) \times 10^{-12}$ [7]. The predictions fit the observed data $\frac{dP}{dt} = (-2.30 \pm 0.22) \times 10^{-12}$ well within the margin of error [8]. For their discovery and subsequent verification of general relativity, Hulse and Taylor were awarded the Nobel Prize in physics in 1993.

5 References

1. Hulse, R.A., and J.H. Taylor. (1975) Discovery of a Pulsar in a Binary System. *Astrophys. J., (Letters)*, **195**, L51-53

2. *Gravitational Wave Detection*, ed. M. Cruise, P. Saulson. (2003) *Proceedings of SPIE*, Volume 4856

3. Peters, P.C., and J. Mathews. (1963) Gravitational Radiation from Point Masses in a Keplerian Orbit. *Phys. Rev.*, **131**, 435.

4. Peters, P.C. (1964) Gravitational Radiation and the Motion of Two Point Masses. *Phys. Rev.*, **136**, B1224.

5. Schutz, B.F. (1985) *A First Course in General Relativity.* Cambridge University Press.

6. Taylor, J.H., R.A. Hulse, L.A. Fowler, G.E. Gullahorn, and J.M. Rankin. (1976) Further Observations of the Binary Pulsar PSR1913+16. *Astrophys. J., (Letters)*, **206**, L53-58.

7. Taylor, J.H., and P.M. McCulloch. (1978) Evidence for the Existence of Gravitational Radiation from Measurements of the Binary Pulsar PSR1913+16. *Ninth Texas Symposium on Relativistic Astrophysics*, ed. J Ehlers, J.J. Perry, M. Walker. New York Academy of Sciences.

8. Taylor, J.H., and J.M. Weisberg (1982) *Astrophys. J.*, **253**, 908.

9. Thorne, K. S. (1978) General Relativistic Astrophysics. *Theoretical Principles in Astrophysics and Relativity*, ed. N.R. Lebovitz, W.H. Reid, P.O. Vandervoort. University of Chicago Press.
10. Wagoner, R.V. (1975) Test for the Existence of Gravitational Radiation. *Astrophys. J., (Letters)*, **196**, L63-65.
11. Wald, R.M. (1984) *General Relativity.* University of Chicago Press.