Initial-Value Problems in General Relativity

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1 Introduction

In this paper the initial-value formulation of general relativity is reviewed. In section (2) domains of dependence, Cauchy surfaces, and globally hyperbolic spacetimes are defined. These constructions are then used in section (3) to describe the initial-value problem for (quasi-)linear diagonal second-order hyperbolic systems. In section (4) the 3+1 ADM decomposition of a globally hyperbolic spacetime is presented. The initial-value problem for the gravitational field is then formulated in section (5).

2 The Causal Structure of Lorentz Manifolds

Let (M, g) be a Lorentz manifold. At every point $p \in M$, the light cone of p has two connected components. A continuous choice of picking out one of these components to be the future, is called a **time-orientation** for M. If M admits a time-orientation, it is said to be **time-orientable**.

Examples of non-time-orientable manifolds can be constructed, in a manner similar to that of the Möbius band. They are pathological, and we shall not be concerned with them. From now on, we assume that M is time-orientable, and that a time-orientation has been specified.

Given a timelike curve γ in M, its tangent will necessarily be always in one of the two connected components of the light-cone. If it is in the future, then γ is said to be **future-directed**.

Let γ be a future-directed timelike curve in M. A point $p \in M$ is called a **future endpoint** of γ, if for every neighbourhood V of p, there exists t_0, such that $\gamma(t) \in V$ whenever $t \geq t_0$. If γ has no future endpoint, it is called **future-inextendible**. Past-directed timelike curves, past endpoints, and past-inextendibility, are defined analogously.

Intuitively, a future-inextendible curve is one that 'goes on forever' into the future. Note that if M has a 'hole', then it is possible to construct a future-inextendible curve, which 'gets arbitrarily close to the hole' as the proper time runs to infinity.

A subset $S \subset M$ is said to be **achronal** if no two of its points can be joined by a timelike curve. For an achronal $S \subset M$, the **future domain of dependence** of S, denoted by $D^+(S)$, is defined to be the set of all points $p \in M$, with the property that every past-directed inextendible timelike curve starting at p intersects S. The **past domain of dependence** of S, denoted by $D^-(S)$, is defined analogously, and the **total domain of dependence** of S is defined to be $D(S) = D^+(S) \cup D^-(S)$.

If $S \subset M$ is achronal and $D(S) = M$, then S is said to be a **Cauchy surface** for M. It can be shown that a Cauchy surface is necessarily an embedded 3-dimensional submanifold.

If M admits a Cauchy surface, it is said to be **globally hyperbolic**. It can be shown that if M is globally hyperbolic, then it can be foliated by a one-parameter family of Cauchy surfaces $\Sigma(\tau)$, and its topology is the product $\Sigma \times \mathbb{R}$. Intuitively, the parameter τ can be thought of as a global time coordinate, and the Cauchy surfaces $\Sigma(\tau)$ can be thought of as 'all of space at fixed time'.

3 The Initial-Value Problem for Matter Fields

We begin with a review of the initial-value problem for a scalar field ϕ in special relativity, whose dynamics is governed by the Klein-Gordon equation, $(\partial^2 - m^2)\phi = 0$. In globally inertial coordinates (t, x, y, z), this reads

$$\frac{\partial^2 \phi}{\partial t^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - m^2 \right) \phi \ . \tag{1}$$

If the values of ϕ and $\partial_t \phi$ are specified on a surface of constant t, then (1) can be solved to uniquely obtain ϕ in all of space-time.

We now describe the generalization of this problem to a globally hyperbolic spacetime M in general relativity. Let Σ be a Cauchy surface for M, with normal vector field n^μ. Suppose that values of ϕ and $n(\phi)$ are specified on Σ. Then $(\nabla^2 - m^2)\phi = 0$ can be solved to uniquely obtain ϕ in all of M.

This result can be generalized to a finite collection of scalar fields ϕ_i, whose dynamics is described by a linear diagonal second-order hyperbolic system

$$\nabla^2\phi_i + A_{ij}(\phi_j) + B_{ij}\phi_j + C_i = 0 \ , \tag{2}$$

where A_{ij} are vector fields and B_{ij}, C_i are scalar fields.

It is possible to generalize even further to a class of systems known as **quasi-linear**, where the metric $g_{\mu\nu}$ is allowed to depend on the unknown fields, and non-linear terms in $\phi_i, \nabla\phi_i$ are allowed:

$$g^{\mu\nu}(\phi_j, \nabla\phi_j)\nabla_\mu\nabla_\nu\phi_i = F_i(\phi_j, \nabla\phi_j) \ . \tag{3}$$

However, only local existence and uniqueness results can be obtained, for initial conditions sufficiently close to a zeroth order solution $\phi_j^{(0)}$.

4 The ADM 3+1 Decomposition

Suppose that M is a spacetime and $\Sigma \subset M$ is a 3-dimensional spacelike submanifold with normal field n^μ. Consider a congruence of timelike geodesics orthogonal to Σ, with tangent field T^μ. The **extrinsic curvature** of Σ is defined by $K_{\mu\nu} = \nabla_\mu T_\nu$. It can be shown that $K_{\mu\nu}$ is symmetric. Let $h_{\mu\nu}$ be the restriction of $g_{\mu\nu}$ to Σ. Then $h_{\mu\nu}$ is a 3-dimensional Riemannian metric, and

$$g_{\mu\nu} = -n_\mu n_\nu + h_{\mu\nu}. \tag{4}$$

A tensor field T on M can be projected down to a tensor field $\pi(T)$ on Σ by

$$\pi(T)^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} = h^{\mu_1}{}_{\rho_1}\cdots h^{\mu_k}{}_{\rho_k} \ \cdot \ h_{\nu_1}{}^{\sigma_1}\cdots h_{\nu_l}{}^{\sigma_l} \ \cdot \ T^{\rho_1\cdots\rho_k}{}_{\sigma_1\cdots\sigma_l} \ . \tag{5}$$

The metric $h_{\mu\nu}$ induces a derivative operator $\tilde{\nabla}$ and Riemann curvature tensor $\tilde{R}_{\mu\nu\rho}{}^\sigma$ on Σ. These are related to the respective quantities on M by

$$\tilde{\nabla}_\lambda T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} = \pi(\nabla_\lambda T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}) \ , \tag{6}$$

$$\tilde{R}_{\mu\nu\rho}{}^\sigma = \pi(R_{\mu\nu\rho}{}^\sigma) - 2K_{[\mu|\rho|}K_{\nu]}{}^\sigma \ , \tag{7}$$

Now consider a globally hyperbolic spacetime M foliated by Cauchy surfaces $\Sigma(\tau)$. Applying the above construction to these surfaces we obtain tensor fields $K_{\mu\nu}(\tau)$ and $h_{\mu\nu}(\tau)$. It can be shown that

$$K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n h_{\mu\nu} \ , \tag{8}$$

where $n^\mu(\tau)$ is now a normal field to $\Sigma(\tau)$ defined on all of M. Thus $K_{\mu\nu}$ can be interpreted as the time-derivative of $h_{\mu\nu}$, so we described the 4-dimensional Lorentz metric as a 3-dimensional Riemannian metric that 'evolves in time'.

The dynamics of $h_{\mu\nu}, K_{\mu\nu}$ is described by Einstein's equation $G_{\mu\nu} = 0$, where for simplicity we consider vacuum space-times (i.e. $T_{\mu\nu} = 0$.) Writing this out in coordinates yields

$$g^{\alpha\beta}(\partial_\alpha\partial_\beta g_{\mu\nu} + \partial_\mu\partial_\nu g_{\alpha\beta} - 2\partial_\beta\partial_{(\nu}g_{\mu)\alpha} + g_{\mu\nu}g^{\rho\sigma}(\partial_\beta\partial_\rho g_{\sigma\alpha} - \partial_\alpha\partial_\beta g_{\rho\sigma})) = F_{\mu\nu} \ , \quad (9)$$

where $F_{\mu\nu}$ is a (non-linear) function of the metric and its first derivatives.

The components of (9) along n^μ contain no second time derivatives, and are thus constraints. They can be written in the form

$$K_{\mu\nu}K^{\mu\nu} - (K^\mu{}_\mu)^2 = \tilde{R} \ , \quad (10)$$

$$\tilde{\nabla}_\nu K^\nu{}_\mu - \tilde{\nabla}_\mu K^\nu{}_\nu = 0 \ . \quad (11)$$

In order to eliminate redundant degrees of freedom, we carry out a gauge transformation to harmonic coordinates in which $\nabla^2 x_\mu = 0$. Equation (9) then becomes

$$g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} = \hat{F}_{\mu\nu} \ , \quad (12)$$

where $\hat{F}_{\mu\nu}$ is again a function (different from $F_{\mu\nu}$) of the metric and its first derivatives. Equation (12) has the form of a quasi-linear diagonal second-order hyperbolic system.

5 The Initial-Value Problem for the Gravitational Field

The initial-value problem for the gravitational field is formulated as follows: Let (Σ, h) be a 3-dimensional orientable Riemannian manifold, and let $K_{\mu\nu}$ be a symmetric tensor field on Σ. Suppose that the constraint equations (10) and (11) are satisfied.

In order to solve the quasi-linear system (12), we need a zeroth-order solution as a starting point. We take this to be flat Minkowski space, denoted by $(\mathbb{R}^4, g^{(0)})$. Σ can be embedded into $\mathbb{R}^4$, such that the restriction of $g^{(0)}_{\mu\nu}$ to Σ is the usual Euclidean 3-dimensional metric. Let n^μ be a normal field to Σ. Initial data $(g_{\mu\nu}, n(g_{\mu\nu}))$ for equation (12) is constructed on Σ as follows:

1. The components of $g_{\mu\nu}$ along Σ coincide with $h_{\mu\nu}$.

2. The components of $n(g_{\mu\nu})$ along Σ are chosen such that the extrinsic curvature of Σ coincides with $K_{\mu\nu}$.

3. The components of $n(g_{\mu\nu})$ along n are chosen such that the harmonic gauge condition $\nabla^2 x_\mu = 0$ is satisfied.

We may assume that this initial data is sufficiently close to the unperturbed metric, such that (12) has a unique solution in some neighbourhood of Σ:

For every $p \in \Sigma$, this can be achieved in a neighbourhood V of p by means of a re-scaling transformation. A solution is then obtained in V. Paracompactness of Σ is used to put these solutions together, to obtain a solution in a neighbourhood of Σ.

Using the Bianchi identity, it can be verified that the solution satisfies both the harmonic gauge condition, and the constraint equations (10)-(11).

In summary, we have obtained a neighbourhood V of Σ, and a solution $g_{\mu\nu}$ of (12) defined in V, such that (V, g) is a globally hyperbolic spacetime, Σ can be embedded into V as a Cauchy surface, the restriction of $g_{\mu\nu}$ to Σ coincides with $h_{\mu\nu}$, and the extrinsic curvature of Σ coincides with $K_{\mu\nu}$.

Finally, a Zorn's lemma argument shows that there exists a solution as described in the previous paragraph, which is moreover maximal, in the sense that any other solution can be isometrically embedded into it.

6 References

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