

The Global Positioning System (GPS) is a wide-spread device that has made the world a much smaller place; people use it everyday to find where they are on the Earth. Interestingly, the GPS is only possible due our understanding of general relativity. It is based upon the idea that if a person simultaneously receives information from four independent locations in space-time via light, then they can solve a system of linear equations to find out what time it is (for them) and where they are (using the fact that the speed of light is the same for all observers); in symbols:

$$c(t - t_i) = |\vec{r} - \vec{r}_i| \quad (1)$$

Relativity enters because we are dealing with objects moving at great speed in different places within a gravitational potential. The system must account for these relativistic effects: a mistake of 100 nanoseconds (10^{-7} s) in the timing measurement results in an error of $(299792458\text{m/s})(10^{-7}\text{s}) \approx 30\text{m}$ in the position measurement. But before diving into the nuts and bolts of the system, a quick glance at how it is set-up is needed to truly be able to model the situation.

The GPS system was developed by the United States Department of Defence, and was in complete operation by 1994. The system consists of twenty-four high-orbit satellites carrying highly-accurate atomic clocks that circle the Earth twice every day and the control stations around the world that monitor them. The control stations keep the satellite's clocks synchronized and correct any deviations to a satellite's orbit that may occur due to unforeseen circumstances. The orbits are configured so that at least four satellites are see-able from any place on the Earth. Each satellite emits a radio signal from space that contains information about what time the satellite has, where it is located in a

reference frame called WGS-84 (or the Earth-Centered, Earth-Fixed reference frame; more details on that to follow), and any important updates that the control stations want to communicate to the GPS receivers. Receivers have an almanac of where each satellite should be at any given time so they can use the information from one satellite to locate other satellites. Another important assumption built into the system is that the Earth is spherical and that the orbits travel along circles (the orbits are mostly circular, but the effects of other massive bodies in the solar system do cause eccentricities: to the accuracy that we are concerned with, ignoring these effects is fine [1]). This is the basic framework that the system works in. The real meat of the problem is setting up the coordinates to be able to solve (1).

The WGS-84 frame is a reference frame where the origin is placed at the centre of the earth, the z-axis is aligned to coincide with the axis of rotation about the earth, and the other two axes are perpendicular to it and rotate with the Earth's spin. This frame is not equivalent to an inertial frame since it is rotating, and thus (1) does not strictly hold in it. This problem is circumvented by using another frame to work with equation (1); we can use coordinate transforms to go between the two frames. The working frame is called the Earth-centered Inertial frame (ECI). Here, the origin is again placed at the centre of the Earth, and the z-axis again coincides with the rotation axis of the Earth, but the x and y axes are fixed and do not rotate with the Earth. The WGS-84 frame is the frame in which we measure the end position.

Relativity enters the scene because the clocks do not all count the same proper time. These proper times must be corrected using general relativity in order to not get

errors that could seriously damage the accuracy of the system. The metric that the makers of GPS use is the Schwarzschild metric:

$$-ds^2 = \left(1 + \frac{2V}{c^2}\right) (c^2 dt^2) - \left(1 + \frac{2V}{c^2}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{4GJ \sin^2 \theta}{rc^3} (cdt) d\phi$$

where this metric is in ECI coordinates, J is the Earth's angular momentum, and V is the Newtonian gravitational potential of the Earth given by solving Poisson's equation for the mass distribution of the Earth:

$$V = -\frac{GM}{r} \left(1 - J_2 \left(\frac{R_E}{r} \right)^2 P_2(\cos \theta) \right)$$

(P_2 is the Legendre polynomial of second degree, J_2 is the quadrupole moment of the Earth). This metric is appropriate for the situation because the impact of the other massive bodies in our solar system cause errors of an order less than 10^{-15} s ([1]) in the time measurement. Similarly, we only need concern ourselves with up to the quadrupole moment of the Earth since the rest of the expansion is smaller by at least a negligible factor of 1000 ([2]), (and in fact we will be able to neglect that term later on). Since the coordinates used by the satellites and receivers are in the WGS-84 frame, we need to transform this metric:

$$t = t', \quad r = r', \quad \phi = \phi' + \omega_E t'$$

where ω_E is the Earth's angular velocity. After the transform, we get:

$$\begin{aligned} -ds^2 = & \left(1 + \frac{2V}{c^2} - \frac{r'^2 \omega_E^2 \sin^2 \theta'}{c^2} + \frac{GJ \omega_E \sin^2 \theta'}{r' c^4} \right) (c^2 dt'^2) - \left(1 + \frac{2V}{c^2} \right)^{-1} dr'^2 - r'^2 d\theta'^2 - r'^2 \sin^2 \theta' d\phi'^2 \\ & - \left(2r'^2 \omega_E \sin^2 \theta' - \frac{4GJ \sin^2 \theta'}{r'^2 c^2} \right) dt' d\phi' \end{aligned}$$

This can be made slightly simpler because the last term is very small compared to the rest of the metric ([2]), so we can safely ignore it. Also computations of the terms of the time-time component of the metric (using values at the equator for estimation):

$$\begin{aligned} 2V/c^2 &= -1.39 \times 10^{-9} \\ -R_E^2 \omega_E^2 \sin^2 \theta' / c^2 &= -2.40 \times 10^{-12} \\ 4GJ \omega_E \sin^2 \theta' / R_E c^4 &= 2.6 \times 10^{-21} \end{aligned}$$

which means we can safely ignore the last term in the component and define an effective potential:

$$\varphi_{\text{eff}} = V - \omega_E^2 r'^2 \sin^2 \theta' / 2$$

making the time-time component of the metric simpler to express. We need to change the time coordinate once more, since right now the time coordinate records what clocks placed infinitely far away from the equator would have. The reference clock standard that the GPS uses is the UTC (USNO): Universal Coordinate Time out of the U.S. Naval Observatory. We can transform to this time scale by using the transform:

$$t'' = t'(1 - 6.9693 \times 10^{-10}) = t'(1 + \xi/c^2)$$

where the constant term ξ is the Terrestrial Time scale as defined by the International Astronomical Union ([1]). Wrapping this all together we find that the metric in the ECI frame is (a term has been dropped from the time-time component since we can neglect things of order $1/c^4$ due to their smallness):

$$-ds^2 = \left(1 + \frac{2(V - \xi)}{c^2}\right) (c^2 dt'^2) - \left(1 + \frac{2V}{c^2}\right)^{-1} dr'^2 - r'^2 d\theta'^2 - r'^2 \sin^2 \theta' d\phi'^2$$

Since the Earth's gravitational field is weak enough, we can approximate the term in radial-radial component by its Taylor expansion:

$$-ds^2 = \left(1 + \frac{2(V - \xi)}{c^2}\right) (c^2 dt'^2) - \left(1 - \frac{2V}{c^2}\right) dr'^2 - r'^2 d\theta'^2 - r'^2 \sin^2 \theta' d\phi'^2$$

Proper time is related to the line element by $ds^2 = -c^2 d\tau^2$. Thus we can write the relationship between the proper time and the coordinate time as:

$$d\tau^2 = \left(1 + 2 \frac{(V - \xi)}{c^2} - \frac{dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)}{c^2 dt^2} \right) dt^2$$

The third term is just the magnitude of the 3-velocity squared, so we get the relation:

$$\int d\tau = \int \sqrt{\left(1 + 2 \frac{(V - \xi)}{c^2} - \frac{v^2}{c^2} \right)} dt$$

where the integrals are taken along the path of the object. Now we can solve for the amount of coordinate time that passes during an interval of a clock's proper time, and use the binomial theorem to simplify:

$$\int dt = \int \left(1 - \frac{(V - \xi)}{c^2} + \frac{v^2}{2c^2} \right) d\tau \quad (2)$$

This is justifiable because the second and third term inside the square root are much smaller than 1.

We now have all we need to make a consistent system of time coordinates to use the GPS. Once we solve (2) and use it to correct the proper time of clocks on the satellites, the signal that a GPS satellite transmits to a receiver on Earth will be a time coordinate consistent with the time coordinates of the UTC (USNO) and thus can be used with equations (1) to find the user's position on the Earth. One last approximation is used in the GPS: since in the V term the quadrupole moment has a $(R_E/r_S)^2$ term, with $r_S = 26562000\text{m}$ and $R_E = 6378137\text{m}$ [2], we can neglect the quadrupole moment. Thus the integral we get has no terms that depend on varying parameters:

$$\int dt = \int \left(1 + \frac{GM_E}{r_s c^2} + \frac{\xi}{c^2} + \frac{v^2}{2c^2} \right) d\tau$$

The constants that we need to integrate this expression are given by [2]:

$$\xi/c^2 = -6.9693 \times 10^{-10}$$

$$v^2/2c^2 = 8.30314 \times 10^{-11}$$

$$GM_E/r_s c^2 = 1.6697 \times 10^{-10}$$

where the speed of the satellite is given by the angular velocity multiplied by the radial distance (each satellite orbits the Earth twice in a day). Thus, in a given day, the satellite must correct its clock by:

$$\Delta t_{day} = (24 - 1.07263 \times 10^{-8}) \text{hrs}$$

where we see that if relativistic effects were not accounted for the system would be inaccurate by about 3 metres and growing each day (since the clock would be consistently slower).

The makers of GPS have used a lot of approximations in their derivation of the time shift due to speed and gravitational effects. Although they are all justified to the accuracy desired by common users, a better system would not use as many approximations and get even better results. The GPS system for the common user is at best accurate to about a radius of 3 meters ([4]), with results for the military being more accurate due to using more and more terms in the multipole expansion of the Earth's gravitational potential.