ON THE PENROSE INEQUALITY

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Abstract

Penrose presented back in 1973 an argument that any part of the spacetime which contains black holes with event horizons of area A has total mass $\sqrt{A/16\pi}$. For the time symmetric case this becomes a problem in Riemannian geometry, and the inequality introduced by Penrose is called the Riemannian Penrose inequality. Outlines of two different proofs are introduced here, as well as background notions.

1. Background on Event horizons and Apparent horizons

The general definition of black holes is the region of spacetime from which no information carrying signal can escape [1]. If one defines the causal past $J^-(Q)$ of a set Q as the set of points that for each point there is a future directed causal curve connecting it with one of the points of Q, then the set of points visible to a distant observer coincides with $J^-(J^+)$ (J^+ is the past null cone)[1]. The boundary of this set denoted by H^+, is called the event horizon. If a space contains no event horizon, then all the events happening in the space can be observed after some time. If an event horizon appears in a space, it means that a black hole has been born and that the only possible way to find out about the events inside the black hole is to cross the horizon and fall into the black hole. According to the Penrose theorem, the event horizons are formed by null geodesics (generators) that have no end point in the future. By following these generators in the future one can see that they never leave the horizon and that they do not intersect each other. When we follow the generators in the past, we are faced with two possibilities: either the generator entered the horizon in a point of intersection with another generator (the caustics - point 2 in the figure below) or it always lied in the horizon. The caustics 1 corresponds to the point where the horizon appears.

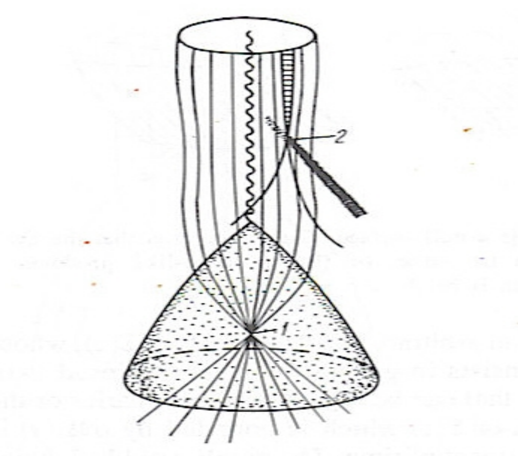

Fig 1: The Penrose theorem

The difficulty in dealing with event horizons is that they cannot be located. If a spacecraft were to go near a black hole, the way it would find the horizon is by finding the point(s) where the space ship could still go back and not be pulled toward the black hole. If we consider a surface in spacetime which emits a shell of light, and the surface area of the shell is decreasing everywhere on the surface, then we call this a trapped surface. The outermost trapped surface is called an apparent horizon.
It is known that apparent horizons, which are easy to locate since they are local, are indicative of black holes. The black hole's event horizon does not coincide with the apparent horizon in general. The event horizon is global, hence to know of its existence or its exact location, a full knowledge of the spacetime's causal structure is required. Therefore, for various applications, apparent horizons are more immediately accessible and practical [2].

2. Mass and Penrose Inequality

In an asymptotically flat spacetime with mass and energy so dense that they collapse under their own gravitation, the formation of a marginally trapped surface H occurs, followed by the formation of a trapped surface. The appearance of a singularity in this region of the spacetime is granted by the singularity theorem. This singularity and the surface H will then be enclosed by an event horizon whose existence is ensured provided the hypothesis of the Cosmic Censorship. As a test of this Cosmic censorship hypothesis, Penrose [3] used the inequality presented below. Though he did not give a mathematical prove for it he noted that if a counterexample that violates this inequality could be found, then one could indicate the failure of the Cosmic Censorship and the existence of a naked singularity (a singularity that is visible). The inequality that we are referring to is:

$$A_H \leq 16\,\pi\,M^2$$

i.e. the total mass of a spacetime containing black holes and event horizons of total area A, cannot exceed $\sqrt{A/16\,\pi}$. In simple terms this means that the area of a black hole is limited by its total mass (for example, for about 100 grams, the area of the respective black hole would be of the order of 10^{-51} cm^2).
The proof of the Riemannian Penrose inequality was first presented in 1997 [4] for one black hole and later [5] for any number of black holes. These proofs rely on a set of common definitions and theorems that we present below.

2.1 Preliminary definitions

Let (M^3, g) be a Riemannian 3-manifold embedded in a 3+1 Lorentz spacetime. Only asymptotically flat at infinity M^3 manifolds are considered. This is equivalent to saying that for some compact set K, $M^3 \backslash K$ is diffeomorphic to $R^3 \backslash B_1(0)$; the metric g is asymptotically approaching the standard flat metric δ_{ij} on R^3 at infinity. Multiple asymptotically flat ends are also allowed, provided that each connected component of $M^3 \backslash K$ must be as described above. Examples of an asymptotically flat manifold is $(R^3; \delta_{ij})$ itself. Other good examples are the conformal metrics $(R^3; u(x)^4\delta_{ij})$, where u(x) approaches a constant sufficiently rapidly at infinity. The limit below ensures M^3 is asymptotically flat

$$m = \frac{1}{16\pi} \lim_{\sigma \to \infty} \int_{S_\sigma} \sum_{i,j} (g_{ij,i}\nu_j - g_{ii,j}\nu_j)\, d\mu$$

where S_σ is the coordinate sphere of radius σ, υ is the unit normal to S_σ, and $d\mu$ is the area element of S_σ in the coordinate chart. The quantity m is called the total mass (or ADM mass) of $(M^3; g)$.
Schoen and Yau [6] have shown that given any $\varepsilon > 0$, it is always possible to perturb an asymptotically flat manifold to become harmonically flat at infinity such that the total mass changes less than ε and the metric changes less than ε pointwise, all while maintaining nonnegative scalar curvature. So it is possible to use harmonically flat manifolds, instead of asymptotically flat if it facilitates calculations.
A flow of 2-surfaces in $(M^3; g)$, in which the surfaces flow in the outward normal direction at a rate equal to the inverse of their mean curvatures at each point, was defined and used by Lang and Wald [7]. The Hawking mass [8] of a surface (which is supposed to estimate the total amount of energy inside the surface) was proved to be nondecreasing under this "inverse mean curvature flow"

$$m_{Hawking}(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_\Sigma H^2\right)$$

(where $|\Sigma|$ is the area of Σ, and H is the mean curvature of Σ in (M^3, g).
The geometry used to prove the inequality is intuitively easy to catch. We define a minimal surface on the manifold to be a surface which is a critical point of the area function with respect to any smooth

variation of the surface. Minimal surfaces have zero mean curvature. The boundary of the union of the open regions bounded by all of the minimal surfaces is indeed a minimal surface itself. Σ_0 is the outer-most of such surfaces [9].

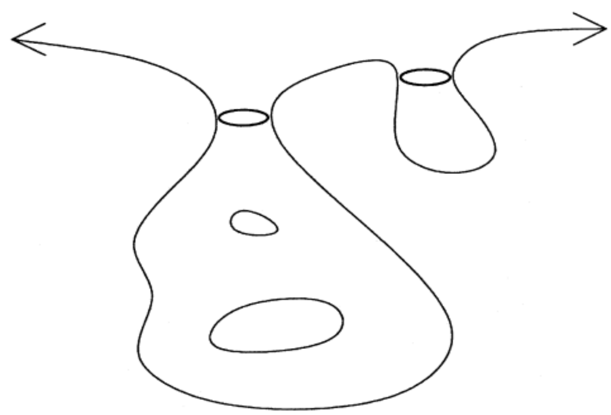

Fig 2: A sketch of a horizon

Since we consider (M^3, g) as the slice t=0, then it has been shown that apparent horizons that intersect with M^3 are the connected components of the outermost minimal surface Σ_0.

We now present a formal statement for the Penrose inequality:

Theorem: The Riemann Penrose Inequality. Let (M^3, g) be a complete, smooth 3-manifold with nonnegative scalar curvature, harmonically flat at infinity, with total mass m, and with an outermost minimal surface Σ_0 of area A_0. Then

$$m \geq \sqrt{\frac{A_0}{16\pi}}$$

and equality holds iff (M^3, g) is isometric to the Schwazschild metric outside the respective outermost minimal surfaces.

A special case of this is known as the Riemannian Penrose inequality. This involves a Riemannian, asymptotically Euclidean 3-manifold with non-negative Ricci scalar, with an outermost minimal surface Σ, which substitutes the apparent horizon. Below we present two different approaches to prove the Riemannian Penrose inequality. For this, another theorem comes handy:

Theorem. The positive mass theorem [6]. Let (M^3; g) be any asymptotically at, complete Riemannian manifold with nonnegative scalar curvature. Then the total mass m $\geq$0, with equality if and only if (M^3; g) is isometric to (R^3; δ).

2.2 Outline of the proof of the Riemannian Penrose inequality

(a) If $\Sigma(t)$ is the surface resulting from inverse mean curvature flow for t beginning with Σ_0 and $\Sigma'(t)$ the outermost minimal area enclosure, we would typically have $\Sigma(t) = \Sigma'(t)$. If this is not the case, then we replace $\Sigma(t)$ by $\Sigma'(t)$ and continue following by inverse mean curvature flow. In this manner the mean curvature of $\Sigma'(t)$ is nonnegative since otherwise it would be enclosed by a surface of less area. The Hawking mass for Σ_0 becomes:

$$m_H = \sqrt{|\Sigma_0|/16\pi}$$

since Σ_0 has zero mean curvature. The Hawking mass is still monotone since $\int \Sigma'(t)H^2 \leq \int \Sigma(t)H^2$ and $|\Sigma'(t)| = |\Sigma(t)|$. The latter is because first, since Σ' is an outermost minimal area we have $|\Sigma'(t)| \leq |\Sigma(t)|$ and we cannot have strict inequality because that implies that $\Sigma(t)$ would have jumped outside $\Sigma'(t)$ at some point in time. Define $\Sigma(t)$ to be a scalar function u(x) in (M^3, g) such that u(x) = 0 on Σ_0 and

$$\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = |\nabla u|$$

where the left-hand side of this equation is the mean curvature of the level sets of u(x) and the right hand side is the reciprocal of the flow rate. To prove the existence of solutions for the above equation Huisken and Ilmanen used the elliptic equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{|\nabla u|^2+\epsilon^2}}\right) = \sqrt{|\nabla u|^2+\epsilon^2}.$$

which yields a weak solution to the first equation when ε tends to zero. These solutions have often flat regions where u(x) is a constant. The level sets $\Sigma(t)$ and u(x) are discontinuous in time and thus we obtain the jumping of $\Sigma(t)$ as noted before. Since the Hawking mass is monotone, with the use of the inverse mean curvature flow method we can prove the Riemann Penrose inequality as well as the positive mass theorem.

(b) In a different context, Bray [5, 10] proved the inequality for several black holes. In order to do this he first defined a continuous, one parameter family of metrics (M^3, g_t), $0 \le t < \infty$, such that it converges to a three dimensional Schwarzschild metric, where g_t have nonnegative scalar curvature and

$A'(t) = 0$
$m'(t) \le 0$ for all $t \ge 0$.

A(t) is the area of the horizon $\Sigma(t)$ of the metric (M^3, g_t), m(t) the total mass of (M^3, g_t), and Σ_0 is the outermost minimal surface of (M^3, g_0), with area A_0. Then, since we chose g_t in such manner that it converges to a Schwarzschild, for which we obtain equality for the Riemannian Penrose inequality, we get

$$m(0) \ge m(\infty) = \sqrt{A(\infty)/16\pi} = \sqrt{A(0)/16\pi}$$

In this way we proved the Riemannian Penrose inequality for the original metric (M^3, g_0). The way Bray found such a family of metrics g_t to satisfy the above conditions, is by introducing another superharmonic function v(x). First he (i) let $g_t = u_t(x)^4 g_0$, then (ii) defined $\Sigma(t)$ to be the outermost minimal area enclosure of Σ_0 in (M^3, g_t), where Σ_0 is the outer minimizing horizon in the original metric. Then (iii) $v_t(x)$ is defined as

$$\begin{cases} \Delta_{g_0} v_t(x) \equiv 0 & \text{outside } \Sigma(t) \\ v_t(x) = 0 & \text{on } \Sigma(t) \\ \lim_{x\to\infty} v_t(x) = -e^{-t} & \end{cases}$$

and $v_t(x) \equiv 0$ inside $\Sigma(t)$. Then (iv) $u_t(x)$ is defined as:

$$u_t(x) = 1 + \int_0^t v_s(x)\,ds$$

Bray tied this representation of $u_t(x)$ to the goal we are trying to reach, by use of three theorems [10,11].

Theorem 1: The relations (i), (ii), (iii) and (iv) define a first order ordinary differential equation in t for $u_t(x)$ having a solution which is Lipschitz in the t variable, class C^1 in the x variable everywhere, and smooth in the x variable outside $\Sigma(t)$. Furthermore, $\Sigma(t)$ is a smooth, strictly outer minimizing horizon in (M^3, g_t) for all $t \geq 0$, and $\Sigma(t_2)$ encloses but does not touch $\Sigma(t_1)$ for all $t_2 > t_1 \geq 0$.
It turns out that the rate of change in g_t is in fact dependent only on g_t itself and $\Sigma(t)$.

Theorem 2. The function A(t) is constant in t, and m(t) is nondecreasing in t, for all $t \geq 0$.
In order to prove that $m'(t) \leq 0$, note first that since the rate of change of g_t does not depend on t, proving $m'(t) \leq 0$ is equivalent to proving $m'(0) \leq 0$.
If we consider $(R^3; u(x)^4\delta_{ij})$, with $u(x) > 0$ that has asymptotics at infinity, we can expand u(x) like $u(x) = a + b/|x| + O(1/|x|^2)$; with the derivatives of the $O(1/|x|^2)$ term being $O(1/|x|^3)$, then the total mass of $(M^3;g)$ is $m = 2ab$ [12].
Here expand $v_0(x)$: $v_0(x) = -1 + c/|x| + O(1/|x|^2)$ for some constant c. Given that the total mass m(t) depends on the rate at which the metric becomes flat, we get $m'(0) = 2(c-m(0))$, i.e. we have to show $c \leq m(0)$.
By removing the region of M^3 inside $\Sigma(0)$ and by reflecting the remainder of (M^3, g_0) through $\Sigma(0)$, one can obtain a manifold whose mass depends on c in the expansion of v. The resulting manifold will now have two resulting flat ends, since (M^3, g_0) has one flat end not included inside $\Sigma(0)$. Similarly, by defining $v_0(x)$ to be the harmonic function which goes to -1 at infinity in the original end and 1 in the reflected end, we have $v_0(x)$ defined on the whole new manifold and zero on $\Sigma(0)$. After compactifying one end of the new manifold, applying the Riemannian positive mass theorem on the new Riemannian manifold $(M^3, (v_0(x)+1)^4 g_0)$ and some calculations we get $m(0) = -4(c-m(0))$ which must be positive by the Riemannian positive mass theorem. Thus $m'(0) = 2(c-m(0)) = -\frac{1}{2} m(0) \leq 0$ (obviously this is merely the general idea, the details would be overwhelming for the scope of this paper. Refer to the appendix for a graphical presentation).
To clarify the assumptions for the area given in the theorem note that since $v_t(x)$ is zero on $\Sigma(t)$, then to first order the metric is not changing on $\Sigma(t)$. Also, the area of $\Sigma(t)$ does not change as it moves outward, since $\Sigma(t)$ is a critical point. From this follows that $A_0(t) = 0$.
This is presented graphically in the figure below. As t increases, $\Sigma(t)$ moves outwards. The fact that the area A(t) remains constant with time, geometrically means that below $\Sigma(t)$ all horizons have the same area. This means that the manifold (M^3, g_0) has a "cylinder-like neck" [10].

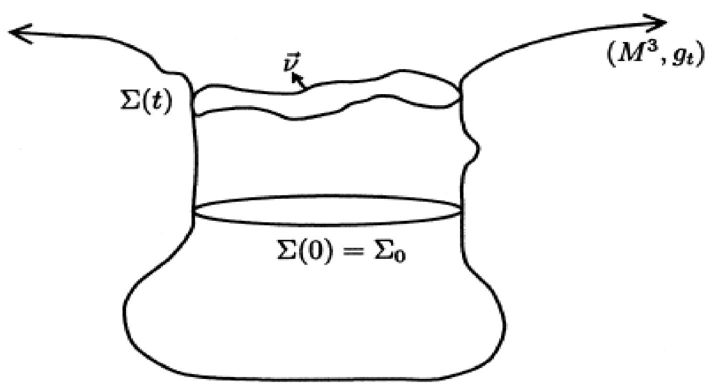

Fig 3. Illustration of constant A(t)

Theorem 3. For sufficiently large t, there exist a diffeomorphism Φt between (M^3, g_0) outside the horizon $\Sigma(t)$ and a fixed Schwarzschild manifold (R3\{0}, s) outside its horizon. Furthermore, for all ε>0, there exists a T such that for all t >T, the metrics gt and Φ*t(s) (when determinig the lengths of unit vectors of (M^3, g_t)) are within ε of each other and the total masses of the two manifold are within ε of each other. Hence,

$$\lim_{t \to \infty} \frac{m(t)}{\sqrt{A(t)}} = \sqrt{\frac{1}{16\pi}}$$

Since Σ(t) by definition encloses any compact set in a finite amount of time, the manifold has zero scalar curvature outside a compact set, u(x) is harmonic outside Σ(t), then one can derive that the scalar curvature of (M^3, g_t) becomes identically zero outside the horizon Σ(t) if (M^3, g_t) is harmonically flat.
The Riemannian Penrose inequality is proved from the three theorems above [11], remembering that asymptotically flat manifolds can be approximated by harmonically flat manifolds. Once the inequality is proved for the latter, it follows for the asymptotically flat case.

3. Conclusion

The Riemannian-Penrose inequality has been proved also for asymptotically hyperbolic metrics [13]. We encounter these metrics when dealing with a negative cosmological constant or when considering hyperboloidal hypersuperfaces in spacetime which are asymptotically flat in isotropic directions.

So far a rigorous proof of the Penrose inequality has been achieved only in two cases: in spherical symmetry and in the time-symmetric case. We already familiarized above with the outlines of the proofs presented by Huisken and Ilmanen [4] for connected Σ and by Bray [10,11], with a totally different method, for arbitrary Σ.

The proof of the inequality in higher dimensions is still an open question.

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APPENDIX

Graphical presentation of the proof that m'(t) ≤ 0 from theorem 2.