The Fate of a Star

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The night sky is littered with a great diversity of stars; there are radiant stars, cold stars, red giants, white dwarfs, black holes, supernovae, and many others, and they come in a variety of sizes and compositions. Moreover, a star goes through many phases through the course of its evolution. In this paper, we examine the possibility of a final equilibrium state for a star as a cold massive body in static, spherically symmetric spacetime. In particular, by treating the body as a perfect fluid and making a few other (natural) assumptions about the distribution of its matter, we find that there is an upper limit on the mass *M* for such solutions: $M \leqslant (4\mathrm{c}^2/9\mathrm{G})\, R$ (where *R* is the radius of the body). After deriving this bound, we will briefly discuss its implications on the possible outcomes of stellar evolution.

Our first task is to derive the upper mass limit – the argument presented here follow closely the discussion in chapter 6.2 of [Wald]. First, we impose the condition that our spacetime is to be static and spherically symmetric, which allows us to restrict our attention to metrics of the form

(1) $$ds^2 = -f(r)\, d\tau^2 + h(r)\, dr^2 + r^2\, d\Omega^2 \quad ,$$

where *f* and *h* are nonnegative functions of *r* alone. The determination of the Einstein tensor $G_{ab} := R_{ab} - (1/2) R\, g_{ab}$ for this metric is a straightforward computation; the off-diagonal entries vanish, while the diagonal entries are given by

(2)
$$G_{\tau\tau} = \frac{h'}{rh^2} + \frac{1-h^{-1}}{r^2}$$
$$G_{rr} = \frac{f'}{rfh} - \frac{1-h^{-1}}{r^2}$$
$$G_{\theta\theta} = G_{\phi\phi} = \frac{1}{2(fh)^{1/2}} \frac{d}{dr}\left[\frac{f'}{(fh)^{1/2}}\right] + \frac{1}{2}\frac{f'}{rfh} - \frac{1}{2}\frac{h'}{rh^2}$$

As mentioned above, the stress-energy tensor of our model is that of a perfect fluid, $T_{ab} = (\rho + P) u_a u_b + P g_{ab}$ where u^a is the unit timelike vector (1,0,0,0), and ρ, P are given functions of *r*. Einstein's equation $G_{ab} = 8\pi T_{ab}$ therefore yields

(3a) $$8\pi\rho = G_{\tau\tau} = \frac{h'}{rh^2} + \frac{1-h^{-1}}{r^2} = \frac{1}{r^2}\left(\frac{rh'}{h^2} + 1 - h^{-1}\right) = \frac{1}{r^2}\frac{d}{dr}\left[r(1-h^{-1})\right]$$

(3b) $$8\pi P = G_{rr} = \frac{f'}{rfh} - \frac{1-h^{-1}}{r^2}$$

(3c) $$8\pi P = G_{\theta\theta} = \frac{1}{2(fh)^{1/2}}\frac{d}{dr}\left[\frac{f'}{(fh)^{1/2}}\right] + \frac{1}{2}\frac{f'}{rfh} - \frac{1}{2}\frac{h'}{rh^2} \quad .$$

We can immediately integrate Eq. (3a):

(4) $$r(1-h^{-1}) = 8\pi \int_0^r \rho(r') r'^2 dr' =: 2m(r) \quad ,$$

and therefore we may write

(5) $$h(r) = \left(1 - \frac{2\mathrm{m}(r)}{r}\right)^{-1} \quad .$$

[Note that the constant of integration in Eq. (4) must necessarily be zero (ie. $m(0) = 0$), in order for the coordinates to hold good as *r* --> 0.]

Now let us suppose, quite naturally, that our perfect fluid is contained in a bounded region, so $\rho(r) = 0$ for $r \geq R$. If we define $M := m(R)$ to be the total mass of the body, then the coordinates inside the star match up smoothly with the vacuum solution of the Schwarzschild metric at the boundary and beyond. This situation is analogous to the Newtonian theory, where the field outside of a spherically symmetric mass distribution is identical to the field given by placing the total mass of the distribution at the origin.

It appears that equation (5) already implies an upper bound for the mass; *h* is not defined if *M* =

$R/2$, and is negative for $M > R/2$. However, the singularity at this point is only an artifact of the coordinate system, and not a genuine singularity; in any case, the bound which we intend to prove is sharper than $M < R/2$.

We now make two further assumptions about the density: first, that $\rho \geqslant 0$, and second, that $d\rho/dr \leqslant 0$. The second assumption accords with the intuition that after sufficiently long times, matter will accumulate in the centre of the body.

Returning to the Einstein equation with a solution for $h(r)$ in hand, we see that taking the difference of equations (3b) and (3c) yields

$$0 = G_{rr} - G_{\theta\theta} = -\frac{1-h^{-1}}{r^2} - \frac{1}{2(fh)^{1/2}}\frac{d}{dr}\left[\frac{f'}{(fh)^{1/2}}\right] + \frac{1}{2}\frac{f'}{rfh} + \frac{1}{2}\frac{h'}{rh^2}$$

(6)

$$= \frac{1}{2}\left(\frac{f'}{r^2(fh)^{1/2}} - \frac{1}{r}\frac{d}{dr}\left[\frac{f'}{(fh)^{1/2}}\right]\right) + (fh)^{1/2}\left(\frac{-2\mathrm{m}}{r^4} + \frac{1}{r^2}\frac{d}{dr}\left[\frac{m}{r}\right]\right)$$

$$= \frac{1}{2}\left((-2)\frac{d}{dr}\left[\frac{1}{r}\frac{f'}{(fh)^{1/2}}\right]\right) + (fh)^{1/2}\left(-\frac{3\mathrm{m}}{r^4} + \frac{m'}{r^3}\right)$$

$$= -\frac{d}{dr}\left[\frac{1}{rh^{1/2}}\frac{d\,f^{1/2}}{dr}\right] + (fh)^{1/2}\left(-\frac{3\mathrm{m}}{r^4} + \frac{m'}{r^3}\right)$$

Now, using the fact that the density is a decreasing function of r, we have

(7) $$\frac{-3\mathrm{m}}{r} + m'(r) = \frac{-3}{r}\int_0^r \rho(r')r'^2\,dr' + \rho(r)r^2 \leqslant -\frac{3}{r}\rho(r)\int_0^r r'^2\,dr' + \rho(r)r^2 = 0 \quad .$$

Therefore, Eq. (6) implies

(8) $$\frac{d}{dr}\left[\frac{1}{rh^{1/2}}\frac{d\,f^{1/2}}{dr}\right] \leqslant 0 \quad ,$$

so that in particular,

$$\frac{1}{rh^{1/2}(r)}\frac{d\ f^{1/2}}{dr}(r) \geqslant \frac{1}{Rh^{1/2}(R)}\frac{d\ f^{1/2}}{dr}(R) \quad . \tag{9}$$

We require our coordinates to be smooth at the boundary *r=R*, so we may compute the derivative of *f* at *R* using the *exterior vacuum solution* of the Schwarzschild metric:

$$\frac{d\ f^{1/2}}{dr}(R) = \frac{d}{dr}\left[\left(1-\frac{2\mathrm{M}}{r}\right)^{1/2}\right]_{r=R} = \frac{M}{R^2\left(1-\frac{2\mathrm{M}}{R}\right)^{1/2}} \quad . \tag{10}$$

Therefore, substituting this into (9), along with our expression for *h(r),* yields

$$\frac{d\ f^{1/2}}{dr} \geqslant r\,h^{1/2}(r)\left(\frac{M\left(1-\frac{2\mathrm{M}}{R}\right)^{1/2}}{R^3\left(1-\frac{2\mathrm{M}}{R}\right)^{1/2}}\right) = \frac{Mr}{R^3(1-2\mathrm{m}/r)^{1/2}} \quad . \tag{11}$$

We now integrate the above expression, this time from 0 to *R:*

$$0 \leqslant f^{1/2}(0) \leqslant f^{1/2}(R) - \frac{M}{R^3}\int_0^R \frac{r'\ dr'}{(1-2\mathrm{m}(r')/r')^{1/2}} = (1-2\mathrm{M}/R)^{1/2} - \frac{M}{R^3}\int_0^R \frac{r'\ dr'}{(1-2\mathrm{m}/r')^{1/2}} \quad , \tag{12}$$

where the first inequality follows since *f* is nonnegative. We wish to find a bound on the integral in the above expression. To do this, note that (7) implies that $d/dr[m(r)/r^3] = -3\mathrm{m}(r)/r^4 + m'(r)/r^3 \leqslant 0$, so in particular $m(r)/r^3 \geqslant m(R)/R^3 = M/R^3$.

Thus, we have that

(13)
$$\begin{aligned}(1-2\mathrm{M}/R)^{1/2} &\geqslant \frac{M}{R^3}\int_0^R \frac{r'\,dr'}{(1-2\mathrm{m}/r')^{1/2}} \\ &\geqslant \frac{M}{R^3}\int_0^R \frac{r'\,dr'}{(1-2\mathrm{M}r'^2/R^3)^{1/2}} \\ &= (1/2)\int_{1-2\mathrm{M}/R}^{1} u^{1/2}\,du \\ &= \frac{1}{2}-\frac{(1-2\mathrm{M}/R)^{1/2}}{2}\end{aligned}$$

Solving for *M* in terms of *R*, we reach our desired conclusion that for a spherically symmetric perfect fluid in static equilibrium confined to a region of radius *R*, the total mass must satisfy

(14)
$$\boxed{M \leqslant \frac{4}{9}R = \frac{4\mathrm{c}^2}{9\mathrm{G}}R} \quad ,$$

with units reinserted in the last expression.

In the special case that the density is uniform (ie. $\rho(r)=\rho_0$ for $r < R$, and $\rho(r)\equiv 0$ for $r \geq R$), the mass, radius and density are related by $M=(4\pi/3)R^3\rho_0$, so the upper mass limit is determined directly by the density:

(15)
$$M \leqslant \frac{4}{9}\frac{c^2}{G\sqrt{3\pi\rho_0}}$$

In particular, the more dense the fluid is, the less of it there needs to be to support equilibrium solutions.

A very interesting feature of the upper mass limit (14) is that the pressure *P(r)* played no role in its derivation; the result is true for dust, radiation, or any other equation of state. However, the pressure does has important physical implications: for stable equilibrium solutions to exist, there must be a force counteracting the attractive force of gravity, which would otherwise cause any matter distribution to eventually collapse.

The upper mass limit has dramatic consequences on stellar evolution, which we shall sketch here briefly. Stars are formed by the accumulation and condensation of gas clouds, which are mostly made up of hydrogen. As the cloud condenses, there will be a time at which the density of gas is sufficiently high for nuclear fusion reactions of hydrogen to occur. At this point the star will begin to radiate, creating helium as a byproduct of the process. As long as there is fuel to burn, so to speak, the gravitation attraction is counteracted by ideal gas pressure on account of the high temperatures in the interior of the star; in this case, the equation of state satisfies $P \propto \rho$. Once the hydrogen has been used up, the star may contract further to a point where the helium will begin to fuse into more complicated atoms, releasing energy and staving off gravitational collapse; this process may repeat itself a few more times, but eventually, once all the possible nuclear fuels have been exhausted, the star begins to cool down. It has been estimated that the Sun will suffer this fate in approximately 5 billion years, while larger, hotter stars can exhaust their supply of fuel in as little as 100 million years [Hawking p.83].

Once the star ceases to radiate energy, it will continue to contract as it cools down. If at any point the inequality (14) fails to be satisfied, then the fate of the star is certain: it will continue to collapse to a black hole. However, if the mass remains within the limit imposed by (14), then equation of state at very high densities can yield pressures sufficient to balance gravitational attraction. For example, at densities in the order of 10^9 g/cm^3, electrons in the star dissociate from their atoms and form a gas, and if the temperature of the star is sufficiently low (ie. so that it is at, or very near, to its lowest energy state), the gas becomes 'degenerate'; the Pauli exclusion principle, which requires electrons that are close to each other to have high momentum, causes such a gas to exert a degeneracy pressure, with equation of state satisfying $P \propto \rho^{5/3}$ [KK p. 219]. For small enough masses, this degeneracy pressure is enough to counteract the attractive force of gravity. Stars that end in this final state are referred to as 'dwarfs' – those which have reached zero temperature are called black dwarfs, while those that are still cooling are called white dwarfs.

If the star is massive enough, then the electron degeneracy pressure may be insufficient to stave off further collapse. However, at densities in the order of 3×10^{12} g / cm^3 (close to the density of the nucleus of an atom), protons and electrons are fused together to form neutrons, which gives rise to a neutron degeneracy pressure, again due to the Pauli exclusion principle [KK p. 198]. Stars which settle into an equilibrium state balanced by gravity and neutron degeneracy pressure are referred to *neutron stars*.

As the central density of a collapsing star approaches nuclear levels and neutron degeneracy pressure begins to assert itself, the collapse slows down near the center of the star, and a shock wave is sent outward towards the boundary. This causes the outer layers of the star to be violently ejected, producing a supernova. It is through this phenomenon that empirical evidence for the existence of neutron stars has been provided; astronomers have found supernovae with small objects at the center that rotate so quickly, they must have densities in the neutron star regime [Wald p. 135]. It should be noted that the mass limit we derived above was for non-rotating objects; however, a rotating object radiates energy, and thus will cease its motion after a sufficiently long period of time.

Of course, if the mass of the star (after burning its fuel and cooling sufficiently), failed to satisfy the upper mass limit (14) at any point, its fate is sealed; regardless of the pressure involved, no equilibrium solutions are possible, and collapse to a black hole is inevitable. On the other hand, equilibrium solutions are possible for stars whose mass is small enough, and observational evidence suggests that they may exist in nature in the form of dwarfs and neutron stars.

References

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