

Rotation in multiple-plane lensing with a strong lens

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1 Introduction

In this paper, we follow the approach of [Pen and Mao(2006)] to demonstrate the presence of rotational effect in multiple-plane lenses.

Gravitational lensing is a direct consequence of the theory of general relativity. In cosmology, it serves as a particularly clean tool for probing the spacetime structure in our universe. In gravitational lens systems, light rays are lensed or deflected by the intervening matter between the source and the observer, much like the deflection due to a familiar optical lens. However, the gauge freedom of the spacetime description makes the interpretation of derived gravitational lensing results a little tricky. Hence, we shall proceed with caution, and concern ourselves only with effects that are observable in a physical system.

While the magnification and image distortion of an extended light source are well-accepted gravitational lensing effects, the image rotation is claimed to be a gauge artifact and unobservable by [Schneider(1997)]. In his paper, Schneider claimed that any multiple-plane lens system can be reduced to an equivalent single-plane gravitational lens with a symmetric amplification matrix, thus yielding no observable rotation. However, [Pen and Mao(2006)] has recently shown that an image rotation is observable in a lensing system with one strong lens and one weak lens. The recent result has attracted some interest within the cosmology community both because of its fundamental existence and its cosmological implications. In this work, we will follow the approach of [Pen and Mao(2006)] to study the rotation in gravitational lenses.

In section 2, we derive the basic equations used in current gravitational lensing research, as well as introduce some general properties of a lensing system. In section 3, we specialize in a system with two lens planes, one weak and one strong in order to demonstrate the presence of image rotation by examining the composed amplification matrix. Such rotational effect is physical, and can be observed using a lensing system with a sufficient number of constraints compared to the degrees of freedom. The mathematical formulation of the measurement for a lensing system with a 3-component source and quadruple images is given in section 4.

2 Gravitational lensing

In this section, we introduce some basic concepts and vocabulary used in the theory of gravitational lensing. We shall introduce the deflection angle and use it to define the lensing potential. We then discuss the lens equation and the amplification matrix, which is the most important tool in the derivation of the rotation angle in section 3. We introduce the convergence and the shear of a gravitational lens for some fixed observer and source positions, as well as mention some observable effects of gravitational lensing. The basic theory covers the single-plane lens scenario.

2.1 Deflection of light by a point mass

We first consider the deflection of a light ray by a single point mass M , which is static and spherically symmetric. The Schwarzschild metric properly describes the gravitational field outside of the Schwarzschild radius at $r_S = 2GM/c^2$.

In the Schwarzschild geometry, assuming that a light ray propagates well outside of the horizon, that is $\xi \gg r_S$ (Assumption 0), General Relativity predicts a deflection of the light ray by the angle

$$\hat{\alpha} = \frac{4GM}{c^2 \xi} \quad (1)$$

as shown in [Wald(1984)]. Here, ξ is the impact parameter, which is the shortest distance between the gravitating mass and the photon trajectory.

2.2 Deflection of light by a distribution of mass

Assuming that the lensing system is in the weak field limit (Assumption 1), we can linearize the field equations of General Relativity. Hence, the deflection angle due to a collection of point masses is simply the vectorial sum of the deflections due to the individual point masses.

As in the analysis of an optical lens, we often compare a lensed light ray to an undeflected one, that is one which propagates in vacuum. We follow the same logic in our analysis of gravitational lenses, and perturb the undeflected light ray, that is one which propagates in a flat spacetime, assuming that the deflection angle is small (Assumption 2). In this work, we adopt the Born approximation (by assumption 2) which allows us to approximate the light ray in the neighbourhood of the deflecting mass by a straight line. We now write the spatial trajectory of an incoming photon as $(\xi_1(\lambda), \xi_2(\lambda), r_3(\lambda))$ in the orthogonal coordinate system such that it propagates along r_3 . Here, λ is an affine parameter. In this parameterization, we obtain under the Born approximation that the projected position (onto a plane perpendicular to the incoming light ray), $\vec{\xi} = (\xi_1(\lambda), \xi_2(\lambda))$, is constant along the photon trajectory. The impact vector of the light ray with trajectory $\vec{r}(\lambda) = (\vec{\xi}, r_3(\lambda))$

due to a mass element dm at $\vec{r}'(\xi'_1, \xi'_2, r'_3)$ is $\vec{\xi} - \vec{\xi}'$ independent of r'_3 . The total deflection angle, as given by [Bartelmann and Schneider(2001)], is

$$\begin{aligned}\hat{\alpha}(\vec{\xi}) &= \frac{4G}{c^2} \sum dm(\xi'_1, \xi'_2, r'_3) \frac{\vec{\xi} - \vec{\xi}'}{|\vec{\xi} - \vec{\xi}'|^2} \\ &= \frac{4G}{c^2} \int d^2\xi' \int dr'_3 \rho(\xi'_1, \xi'_2, r'_3) \frac{\vec{\xi} - \vec{\xi}'}{|\vec{\xi} - \vec{\xi}'|^2}\end{aligned}\quad (2)$$

$$= \frac{4G}{c^2} \int d^2\xi' \Sigma(\vec{\xi}') \frac{\vec{\xi} - \vec{\xi}'}{|\vec{\xi} - \vec{\xi}'|^2} \quad (3)$$

where

$$\Sigma(\vec{\xi}) \equiv \int dr_3 \rho(\xi_1, \xi_2, r_3). \quad (4)$$

Here, we assumed that the angular size of the image is very small (Assumption 3, and it is typically very small with values of order 1 arc sec $\approx 2.9 \times 10^{-6}$ for cluster lensing) so that the coordinate r_3 for incoming light rays from different parts of the image is roughly aligned. This superposition of deflection angles is valid for a mass distribution with a finite thickness as long as the deviation of the actual light path from a straight line within the mass distribution is small compared to the scale on which the mass distribution changes significantly (Born approximation revisited). In deriving Equation 3, we also used the assumption that the angular size of the source is small so that we can perform planar integration after projecting the mass distribution along the radial coordinate.

2.3 Lens equation, deflection potential

Here, we introduce the lens equation which relates the true position of the source to its observed position on the sky. We deal with a geometrically-thin matter distribution at a time (Assumption 4). In this case, the distance to the lens is well-defined. We also assume that the source is small in the angular size and the thickness (Assumption 5). Then, as depicted in Figure 1, we can define the source and lens planes as planes perpendicular to a straight line from the observer to the lens, which we call the optical axis, containing the source and the lens respectively. Because the deflection angle is typically small (Approximation 2), the optical axis roughly align with the coordinate r_3 introduced previously. Where $\vec{\eta}$ represents the two-dimensional position of the source on the source plane, we see from Figure 1 that

$$\vec{\eta} = \frac{D_s}{D_d} \vec{\xi} - D_d \hat{\alpha}(\vec{\xi}). \quad (5)$$

Using the angular coordinates $\vec{\beta} = \vec{\eta}/D_s$ and $\vec{\theta} = \vec{\xi}/D_d$, while defining the rescaled deflection

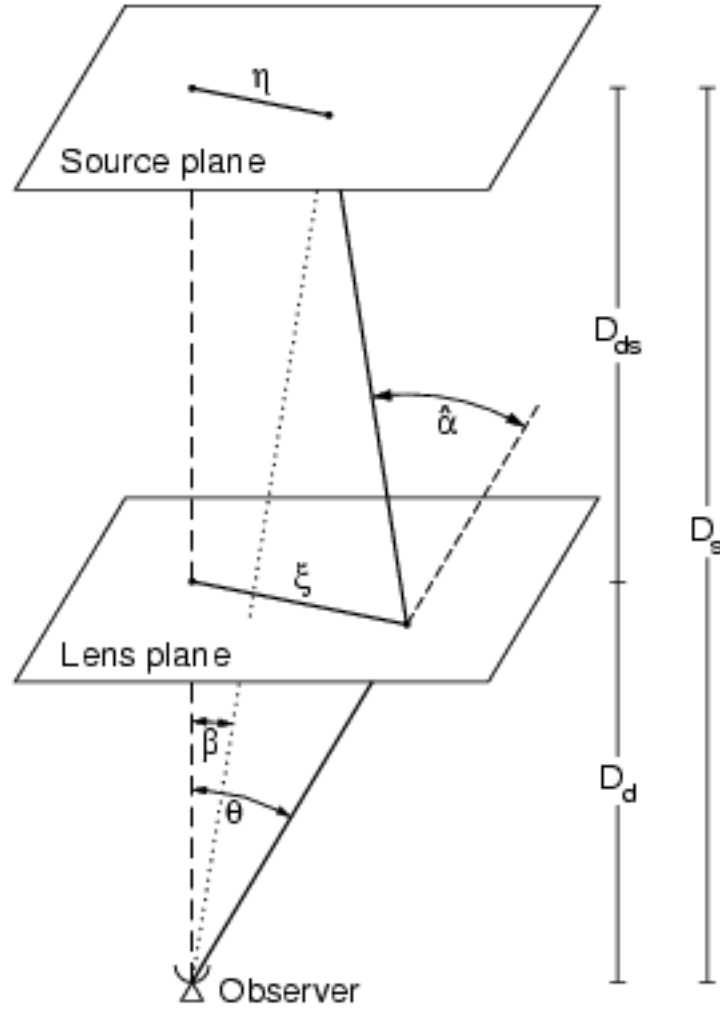


Figure 1: Diagram verbatim from [Bartelmann and Schneider(2001)]. Sketch of a typical gravitational lens system with an extended source and a single-plane lens

angle $\vec{\alpha}(\vec{\theta}) = (D_{ds}/D_s)\hat{\alpha}(D_d\vec{\theta})$, we obtain the reduced form of Equation 5

$$\vec{\beta} = \vec{\theta} - \frac{D_{ds}}{D_s} \hat{\alpha}(D_d\vec{\theta}) \equiv \vec{\theta} - \vec{\alpha}(\vec{\theta}) . \quad (6)$$

As shown in [Schneider et al.(1992)Schneider, Ehlers, and Falco], if the surface density $\Sigma(\vec{\xi})$ is greater than the critical surface mass density

$$\Sigma_{cr} = \frac{c^2}{4\pi G} \frac{D_s}{D_d D_{ds}} \quad (7)$$

at any point of the lens plane, then Equation 6 has multiple solutions for some source position $\vec{\beta}$, therefore producing multiple images for a source at $\vec{\beta}$. A lens system where multiple images of the

same source are observed is called a *strong* lens system; otherwise, the lens is *weak*.

Defining the dimensionless surface mass density

$$\kappa(\vec{\theta}) = \frac{\Sigma(D_d \vec{\theta})}{\Sigma_{\text{cr}}} \quad \text{with} \quad \Sigma_{\text{cr}} = \frac{c^2}{4\pi G} \frac{D_s}{D_d D_{\text{ds}}}, \quad (8)$$

the scaled deflection angle, as given in [Bartelmann and Schneider(2001)], can be expressed as

$$\vec{\alpha}(\vec{\theta}) = \frac{1}{\pi} \int d^2\theta' \kappa(\vec{\theta}') \frac{\vec{\theta} - \vec{\theta}'}{|\vec{\theta} - \vec{\theta}'|^2}. \quad (9)$$

Since $\vec{\alpha}$ is curl-free, we can write it as $\vec{\alpha} = \nabla\psi$ where

$$\psi(\vec{\theta}) = \frac{1}{\pi} \int d^2\theta' \kappa(\vec{\theta}') \ln|\vec{\theta} - \vec{\theta}'|, \quad (10)$$

and ψ satisfies the Poisson equation $\nabla^2\psi(\vec{\theta}) = 2\kappa(\vec{\theta})$

2.4 Amplification Matrix

The important tool for our analysis, the amplification matrix, is defined as the Jacobian matrix of source coordinates with respect to the image coordinates

$$\mathcal{A}(\vec{\theta}) = \frac{\partial \vec{\beta}}{\partial \vec{\theta}} = \left(\delta_{ij} - \frac{\partial^2 \psi(\vec{\theta})}{\partial \theta_i \partial \theta_j} \right) = \begin{pmatrix} 1 - \kappa - \gamma_1 & -\gamma_2 \\ -\gamma_2 & 1 - \kappa + \gamma_1 \end{pmatrix}. \quad (11)$$

Here, we have introduced the complex or spin-2 shear $\gamma = \gamma_1 + i\gamma_2 = |\gamma|e^{2i\phi}$.

3 Rotation in gravitational lenses

In this section, we consider lensing with multiple planes with one and only one of which being a strong lens. The requirement of the present analysis that only one lensing plane can be strong is not much of a constraint in cosmological scenario since the the probability for the alignment of two strong lenses along a line of sight is very small. Since the amplification matrix of two weak lenses simply superpose, we can reduce any system with a multiple planes of weak lenses at different distances to the observer to one with a single weak lensing plane at some new distance. This effective weak lensing plane can in principle lie in front of, on the same plane as or behind the strong lens. However, a weak lens plane coinciding with the strong lens can be absorbed into the strong lens, and contributes no rotation. Hence, we only need to consider the other two cases.

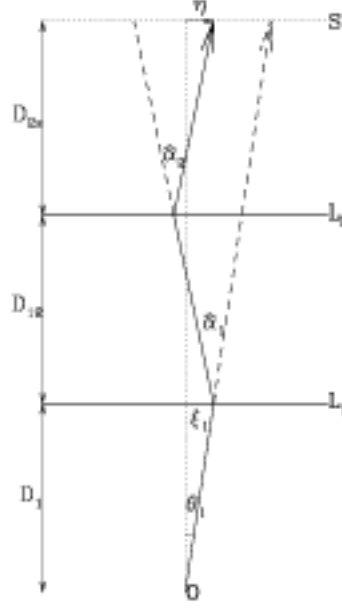


Figure 2: The lensing geometry of a two-plane lens. L_1 is a strong lens, and L_2 weak. η and ξ_1 are the source and image positions respectively. x_1 is the angular size of the image. The deflection angles, $\hat{\alpha}_1$ and $\hat{\alpha}_2$, themselves are not observable; however, their variations over an extended image can be detected with the amplification metric.

Only the derivation of the overall amplification matrix for the case with the strong lens in front of the weak lens is shown here. The calculation for the other case is nearly identical and self-evident.

The 2-plane lens shown in Figure 2 consists of a strong lens L_1 in the foreground and a weak lens L_2 in the background between the observer O and the source plane S . We denote the angular diameter distance from the observer to the first lens by D_1 , that from the first lens to the second lens by D_{12} , that from the second lens to the source by D_{2s} , and so on. The convergence and shear for the strong lens are denoted by κ and γ ; whereas, those for the weak lens are denoted by κ' and γ' . The deflection angles at the first and second planes are denoted by $\hat{\alpha}_1$ and $\hat{\alpha}_2$ respective.

Since the system contains a strong lens, it produces multiple images of the source. We assume that different images pass through different parts of the strong lens with different amplification matrices, as well as different parts of the weak lens with different values of the convergence and the shear. For each image, we need to calculate the amplification matrix in a neighbourhood of the image at each of the lens planes. The amplification matrix is related to the second derivative of the potential or the values of the convergence and the shear. Our first task is to define a lensing potential and to obtain an expression for the convergence and for the shear.

Since a gravitational lensing potential defined on the lens plane is generally assumed to have continuous second derivatives (Assumption ??), we can carry out Taylor expansion around any

point $\vec{\theta}_0$ for small $\delta\vec{\theta} = \vec{\theta} - \vec{\theta}_0$, and we get

$$\delta\psi(\vec{\theta}) = \psi(\vec{\theta}_0) + \left. \frac{\partial\psi}{\partial\theta^i} \right|_{\vec{\theta}_0} \theta^i + \frac{1}{2} \left. \frac{\partial^2\psi}{\partial\theta^i\partial\theta^j} \right|_{\vec{\theta}_0} \theta^i\theta^j + O(x^3). \quad (12)$$

We may assume that the source is sufficiently small that the higher order terms are not important. On the other hand, the constant term and the first order terms with constant coefficients do not contribute to the amplification matrix as the entries are the second derivatives of the potential. There, we adopt the quadratic potential

$$\psi = a\theta_1^2 + 2b\theta_1\theta_2 + c\theta_2^2 \quad (13)$$

as in [Pen and Mao(2006)] where a , b and c are constants. Such potential is the most general form that leads to constancy of κ and γ .

From Equation 11, we see that

$$\frac{\partial^2\psi(\vec{\theta})}{\partial\theta_i\partial\theta_j} = \begin{pmatrix} 2a & 2b \\ 2b & 2c \end{pmatrix} = \begin{pmatrix} \kappa + \gamma_1 & \gamma_2 \\ \gamma_2 & \kappa - \gamma_1 \end{pmatrix}. \quad (14)$$

Hence, for a general homogeneous, isotropic lens with potential given by Equation 13, the convergence and the shear are given by

$$\kappa = a + c, \quad \gamma_1 = a - c, \quad \text{and} \quad \gamma_2 = 2b. \quad (15)$$

In the remainder of the section, we shall derive the rotational effect of an image in the lensing geometry depicted in 2. This is achieved by first writing down the lens equation and then calculating the amplification matrix for the two-plane system in terms of the amplification matrices for the individual lens planes.

Since the deflection angles are very small ($\sim 10^{-4}$ rad), we expand to first order in the deflection angles, as described in section 2 and [Schneider et al.(1992)Schneider, Ehlers, and Falco], to get

$$\vec{\eta} = \frac{D_s}{D_1} \vec{\xi}_1 - D_{1s} \hat{\alpha}_1(\vec{\xi}_1) - D_{2s} \hat{\alpha}_2(\vec{\xi}_2), \quad (16)$$

$$\vec{\xi}_2 = \frac{D_2}{D_1} \vec{\xi}_1 - D_{12} \hat{\alpha}_1(\vec{\xi}_1) \quad (17)$$

where $\vec{\eta}$, $\vec{\xi}_1$ and $\vec{\xi}_2$ are the position vectors in the source plane, the first and the second lens planes respectively. Following [Schneider et al.(1992)Schneider, Ehlers, and Falco], by defining

two rescaled deflection angles

$$\tilde{\alpha}_1 = \frac{D_{1s}}{D_s} \hat{\alpha}_1, \quad \tilde{\alpha}_2 = \frac{D_{2s}}{D_s} \hat{\alpha}_2, \quad (18)$$

and angular positions

$$\tilde{\beta} = \frac{\tilde{\eta}}{D_s}, \quad \tilde{\theta}_1 = \frac{\tilde{\xi}_1}{D_1}, \quad \tilde{\theta}_2 = \frac{\tilde{\xi}_2}{D_2}, \quad (19)$$

we obtain the simple form of the lens equation 17

$$\tilde{\beta} = \tilde{\theta}_1 - \alpha_1(\tilde{\theta}_1) - \alpha_2(\tilde{\theta}_2) \quad \text{and} \quad \tilde{\theta}_2 = \tilde{\theta}_1 - \beta_{12}\alpha_1(\tilde{\theta}_1) \quad (20)$$

where $\beta_{12} = D_{12}D_s/(D_2D_{1s})$.

For the two-plane lensing system, the combined amplification matrix can be read off from Equation 20

$$\mathcal{A} = \frac{\partial \tilde{\beta}}{\partial \tilde{\theta}_1} = I - \mathbf{U}_1 - \mathbf{U}_2 + \beta_{12}\mathbf{U}_2\mathbf{U}_1. \quad (21)$$

Here, I is the identity matrix, and

$$\mathbf{U}_1 = \partial \tilde{\alpha}_1 / \partial \tilde{\theta}_1 = I - \mathcal{A}_1 = \begin{pmatrix} \kappa + \gamma_1 & \gamma_2 \\ \gamma_2 & \kappa - \gamma_1 \end{pmatrix} \quad \text{and} \quad \mathbf{U}_2 = \partial \tilde{\alpha}_2 / \partial \tilde{\theta}_2 = I - \mathcal{A}_2 = \begin{pmatrix} \kappa' + \gamma'_1 & \gamma'_2 \\ \gamma'_2 & \kappa' - \gamma'_1 \end{pmatrix} \quad (22)$$

where \mathcal{A}_i is the amplification of the i^{th} plane *in the absence of the other lens plane*, so are the convergences and the shears. We see from Equation 21 that all terms are symmetric except for $\mathbf{U}_2\mathbf{U}_1$ since the product of two symmetric matrices may not be symmetric if they do not share the same set of eigen vectors. This asymmetry in the amplification \mathcal{A} is responsible for the image rotation as we will see later in this section.

Let us first compute the product term

$$\mathbf{U}_2\mathbf{U}_1 = \begin{pmatrix} (\kappa' + \gamma'_1)(\kappa + \gamma_1) + \gamma'_2\gamma_2 & (\kappa' + \gamma'_1)\gamma_2 + \gamma'_2(\kappa - \gamma_1) \\ \gamma'_2(\kappa + \gamma_1) + (\kappa' - \gamma'_1)\gamma_2 & \gamma'_2\gamma_2 + (\kappa' - \gamma'_1)(\kappa - \gamma_1) \end{pmatrix} \sim \begin{pmatrix} 0 & \gamma'_1\gamma_2 - \gamma'_2\gamma_1 \\ -(\gamma'_1\gamma_2 - \gamma'_2\gamma_1) & 0 \end{pmatrix}. \quad (23)$$

Here, the approximation comes from assuming that the weak lens strength is much weaker than that of the strong lens (assumption). We drop the first order terms in κ' and γ' which contribute to the magnification and distortion of the image since zeroth order terms are present in the amplification matrix and dominate the magnification and distortion effects. However, we keep the first order terms which contribute to the rotational effect, ie. the antisymmetric components, as this is not present in the zeroth order.

Applying the same approximation scheme on Equation 21, the term \mathbf{U}_2 becomes unimportant

and we get

$$\mathcal{A} \sim \begin{pmatrix} 1 - \kappa - \gamma_1 & -\gamma_2 + \beta_{12}(\gamma'_1\gamma_2 - \gamma'_2\gamma_1) \\ -\gamma_2 - \beta_{12}(\gamma'_1\gamma_2 - \gamma'_2\gamma_1) & 1 - \kappa + \gamma_1 \end{pmatrix} = \begin{pmatrix} 1 - \kappa - \gamma_1 & -\gamma_2 + \omega \\ -\gamma_2 - \omega & 1 - \kappa + \gamma_1 \end{pmatrix} \quad (24)$$

where the antisymmetric component is expressed as $\omega = \beta_{12}(\gamma'_1\gamma_2 - \gamma'_2\gamma_1)$. We then recast the amplification matrix given by Equation 24 in a more enlightening form by factoring it as a product of a pure rescaling (symmetric) and a pure rotation $\mathcal{A} = \mathcal{A}_s \mathbf{R}(\phi)$ where

$$\mathbf{R}(\phi) = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}. \quad (25)$$

The rotation angle ϕ is calculated in [Pen and Mao(2006)] to be given by

$$\tan \phi = -\frac{\omega}{1 - \kappa}. \quad (26)$$

The rotation of a single image is not observable as the properties of the distant source is generally not known and that the absolute orientation of the source is not well-defined due to the gauge freedom in General Relativity. In the next section, we proceed to describe how the presence of image rotation can be detected in a physical gravitational lens system by measuring the relative rotation.

4 Measurement of rotation

The simplest lensing system where the effect of rotation can be measured is one where a 3-component source is lensed by a strong lens in combination with one or more weak lenses in the foreground or background to form 4 images, that is, 4 copies of the 3 source components. Let us denote the positions of the 3 source components on the source plane by $P^1 = (P_x^1, P_y^1)$, P^2 and P^3 respectively, and that of the 4 images by A, B, C and D . Without loss of generality, we let P^3 sit at the origin. Furthermore, for image A , we define the apparent positions of P^1 and P^2 relative to that of P^3 as A^1 and A^2 . Then we have

$$P^1 = \mathcal{A}_A A^1 \quad (27)$$

and

$$P^2 = \mathcal{A}_A A^2 \quad (28)$$

where \mathcal{A}_A is the amplification matrix for the image A . Similar pairs of equations can be obtained for the other 3 images B, C and D involving the relative image positions of the first and second

components to the third component, as well as the corresponding amplification matrix. Recall that the multiple-plane lens is assumed to be homogeneous over the size of the source. Hence, κ and γ are constant around the patches on the lens planes where the light forming an image with 3 components passes through, and the 3 components share the same amplification matrix. Writing the two column vectors P^1 and P^2 as a 2×2 matrix

$$\mathbf{P} = \begin{pmatrix} p_x^1 & p_x^2 \\ p_y^1 & p_y^2 \end{pmatrix}, \quad (29)$$

and combining the image position vectors in a similar manner, we obtain a set of homogeneous linear equations

$$\mathcal{A}_A A - P = \mathbf{0} \quad (30)$$

$$\mathcal{A}_B B - P = \mathbf{0} \quad (31)$$

$$\mathcal{A}_C C - P = \mathbf{0} \quad (32)$$

$$\mathcal{A}_D D - P = \mathbf{0}. \quad (33)$$

Here, A, B, C and D are determined by observations whereas $\mathcal{A}_A, \mathcal{A}_B, \mathcal{A}_C, \mathcal{A}_D$ and P are to be solved for. Since the amplification matrices are symmetric, they each have 3 independent components. We can also fix $P_x^2 = 1$ because of a length scale degeneracy of the system and since we are interested in the angular measure of image rotation. Therefore, we have an over-determined system of 16 equations and 15 unknowns, which would enable us to detect a linear combination of the rotation angles for the various images, hence demonstrating the existence of image rotations. If the rotational effect is indeed present as predicted by the theory presented here and in [Pen and Mao(2006)], we can distinguish a multiple-plane lens system from a single-plane one. Such distinction is of great interest to the cosmology community.

5 Conclusion

In summary, we followed the derivation in [Pen and Mao(2006)] to derive the image rotation angle of a multiple-plane lens system. The rotation of a single image is not observable as the properties of the distant source is generally not known and that the absolute orientation of the source is not well-defined due to the gauge freedom in General Relativity. However, in contrary to the claim made in [Schneider(1997)], in a lensing system with multiple images and enough constraints compared to the degrees of freedom, one can measure a linear combination of the rotation angles for the different images, hence confirming the existence of image rotation. The existence of image rotation would

provide a mean to distinguish multiple-plane lensing from single-plane lensing.

References

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