EINSTEIN METRICS AND CONFORMAL INFINITY

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ABSTRACT. Conformal compactness and conformal infinity are defined and discussed. Einstein metrics are defined. Some examples are presented. The problem of uniqueness and existence for Einstein metrics with a given conformal infinity is posed. Three recent results of Michael Anderson are stated. This review is intended for an audience as knowledgable as the author was when he started to read on the topic, with a few reminders of background material thrown in where it does no harm to the flow.

1. INTRODUCTION

Recent trends in String Theory motivate the examination of certain kinds of geometrical boundary value problems [7]. There are conjectured correspondences between field theories on Anti-de Sitter spaces and *conformal field theories* on spaces of one dimension lower — that is, on their boundaries. Whatever is meant by this, it poses some problems to mathematicians. In particular, it asks what is the relationship between Einstein metrics on a manifold and conformal structure at infinity.

For a PDE governing a field on all of $\mathbb{R}^n$, the imposition of a boundary condition can be achieved by specifying asymptotic values of the field, say, with respect to some homotopy $h : S^{n-1} \times [0, \infty) \to \mathbb{R}^n$. Questions about the existence and uniqueness of solutions can then be addressed.

However, where the metric structure itself is at issue, this is not a useful approach. It is preferable to more concretely describe the infinity on which the boundary condition sits.

2. CONFORMAL INFINITY

For an oriented manifold with boundary $\overline{M}$, its interior M, and a metric g on M only, we say that M is conformally compact if there exists a smooth, non-negative function $\rho : \overline{M} \to \mathbb{R}$, with $d\rho \neq 0$ on ∂M and $\rho^{-1}(0) = \partial M$, such that $\rho^2 g$ extends continuously to a metric γ on ∂M. Call the metric $\overline{g} = \rho^2 g + \gamma$ on $\overline{M}$ the *compactification* of g. ρ is called the defining function for the compactification. [6]

This description includes manifolds where the boundary is taken to be at infinity. For example, take M to be the open unit ball centred at 0 in $\mathbb{R}^3$. Then ∂M is the unit sphere. If $\overline{M}$ is parameterized in $\mathbb{R}^3$ by radial coordinate r and angular coordinate Ω, let

$$g = \frac{dr^2}{(1-r^2)^2} + \frac{1}{4}\log^2\left(\frac{1+r}{1-r}\right) d\Omega^2.$$

This (M, g) is isometric to $\mathbb{R}^3$ with the Euclidean metric, under the map $(r, \Omega) \to \left(\frac{1}{2}\log\left(\frac{1+r}{1-r}\right), \Omega\right)$. If $\rho = 1 - r^2$, then

$$\rho^2 g = dr + \frac{1}{4}(1-r^2)^2 \log^2\left(\frac{1+r}{1-r}\right) d\Omega^2 = \overline{g},$$

which extends smoothly to $\overline{M}$ and induces the trivial metric $\gamma = 0$ on ∂M. We conclude that Euclidean $\mathbb{R}^3$ is conformally compact, and it clearly follows for higher dimensional Euclidean space.

This property of conformal compactness determines, among other things, that the manifold does not twist or skew to a pathological degree as one traces outward toward infinity — that, in some sense, it merely "gets larger." Most importantly, it gives an induced metric γ on ∂M. However, γ is by no means unique. In general, many defining functions will compactify (M, g). For example, if ρ is one such function, and τ is any smooth, nonzero function on $\overline{M}$, then $\tau\rho$ is also a defining function for g, and will give rise to a different γ.

If γ and γ' are two such induced metrics on ∂M, then they are in the same conformal class. That is, there is a diffeomorphism $c : \partial M \to \partial M$ and a function $\sigma : \partial M \to \mathbb{R}$ such that $c(\gamma) = \sigma\gamma'$ (where γ is pushed forward as a tensor field). This is the same as saying that, for any two tangent vectors v and w to a point $x \in \partial M$, their cosine w.r.t. γ is unchanged under the map c w.r.t. γ':

$$\frac{v\gamma w}{|v|_\gamma |w|_\gamma} = \frac{c(v)\gamma' c(w)}{|v|_{\gamma'} |w|_{\gamma'}}$$

In fact, γ and γ' will satisfy this relationship for the identity map, since at any point they differ only by a scalar multiple.

We denote by $[\gamma]$ the conformal equivalence class of metrics on ∂M to which γ belongs. In the example, $[\gamma] = 0$, the trivial class. As we have established, there is a unique $[\gamma]$ for each conformally compact metric g on M; however, there is certainly not a unique g for each $[\gamma]$. In fact, an arbitrary deformation of g on a compact set can be made, without affecting the conformal structure at infinity.

We can go further, denoting by $[g]$ the equivalence class of metrics g' on M related to g by orientation-preserving diffeomorphisms (which push forward g' to g) - all such g' describe the same geometry as g. In fact, all such g' have the same conformal infinity as g.

In both cases, the space of equivalence classes can be imbued with a quotient topology (i.e. $\gamma_i \to \gamma \Rightarrow [\gamma_i] \to [\gamma]$) with respect to whatever topology on the function spaces is most useful, typically that which arises from a Hölder or Sobolev norm. However, $[\gamma]$ still does not have a unique $[g]$, no matter by how much we restrict the class of metrics function-analytically.

3. Einstein Metrics

If we restrict the class of metrics g on M to those satisfying a partial differential equation, then there is hope that uniqueness can be attained. Recall Einstein's field equation with a cosmological constant:

$$Ric_g - \frac{1}{2}sg + \Lambda g = 8\pi GT$$

Here, Ric_g is the Ricci curvature tensor associated with the metric g (in Einstein notation under a coordinate chart, it is denoted $R_{\mu\nu}$ and equal to the contraction

${R^\lambda}_{\mu\lambda\nu}$ of the Riemann curvature tensor), s is the scalar curvature (also denoted $R = g^{\mu\nu}R_{\mu\nu}$, where $g^{\mu\nu}$ is the inverse of the metric tensor), Λ is the cosmological constant, G is the gravitational constant and T is the stress-energy tensor. [5]

We say a metric g is an *Einstein metric* if it is (nontrivially) proportional to the Ricci curvature everywhere, that is, it is a vacuum solution of the Einstein equation with nonzero cosmological constant. In particular, the theory treats the case normalized such that if $\dim M = n+1$ then

$$Ric_g = -ng.$$

Einstein metrics are critical points of the Einstein-Hilbert action

$$\mathcal{S} = \int_M (s - 2\Lambda) dV,$$

with the caveat that for a nonzero cosmological constant, this integral is typically infinite.

Disregarding the particulars of the conformal infinity, for a compact manifold with boundary, and a metric specified on the boundary, this is already an underdetermined problem in PDEs — while both g and Ric_g have the same number of degrees of freedom at any one point, ten in 4-D for example, and we therefore might hope to have, up to the usual function-analytic difficulties, a well-posed problem, the Bianchi differential identity on Ric_g reduces the effective degrees of freedom of Ric_g. We expect an underdetermination, however, since, as in the infinite case, a diffeomorphism — that is, an isometry — on the interior will describe the same geometry. Diffeomorphism invariance, suitably interpreted, is one of the symmetries of Einstein's equation.

For any C^2 conformally compact Einstein metric g with defining function ρ, $|K_g + 1| = O(\rho^2)$, where K_g is the sectional curvature (the product of curvatures in principal directions); that is, $K_g \to -1$ near infinity. (See [2].) We call such metrics asymptotically hyperbolic (AH), indicating that their local geometry approaches that of hyperbolic space near infinity. However, their global structure may be quite varied.

4. Geodesic Compactification

An advantage of the non-uniqueness of the defining function is that one may be chosen to match particular challenges. Often the most natural one is defined in the following way: if a function t satisfies

$$t(x) = dist_{\overline{g}}(x, \partial M)$$

then the compactification $\overline{g} = t^2 g$ and the defining function t are said to be *geodesic.* [2]

The *cut locus* of a closed set S in a manifold M is the closure of the set of points in M which have at least two shortest paths to S. The existence of a geodesic compactification of M requires a weakening of our definitions; t will not be differentiable on the cut locus of ∂M. However, t will be continuous on the cut locus and smooth elsewhere, and the cut locus is bounded away from ∂M, therefore the non-differentiability of t is restricted away from the region of interest.

We construct an example on the unit ball in $\mathbb{R}^3$. As before, it is parameterized by radial coordinate r and angular coordinate Ω. Let

$$g = \frac{dr^2}{(1-r)^2} + \log^2(1-r)d\Omega^2$$

which, of course, is merely continuous at $r = 0$. Then the ball is isometric to Euclidean space under the map $(r, \Omega) \to (-\log(1-r), \Omega)$. If $t = 1-r$ is the defining function, then clearly t is the distance from the boundary w.r.t the compactification

$$\overline{g} = t^2 g = dr^2 + (1-r)^2 \log^2(1-r)d\Omega^2$$

Both examples so far are flat, therefore neither is an Einstein metric in our sense.

We note that for a geodesic defining function t, the integral curves of $\overline{\nabla}t$, where $\overline{\nabla}$ indicates gradient w.r.t. $\overline{g}$, are geodesics (with respect to both $\overline{g}$ and g, it turns out). This is, of course, the motivation for applying the term *geodesic* in this case.

If $\overline{g}$ is a geodesic compactification, and t its defining function, then, away from the cut locus of ∂M,

$$\overline{g} = dt^2 + g_t$$

where, for each t, g_t is a metric, of dimension that of ∂M, which is transverse to dt^2. Since g_t is a function from $\mathbb{R}$ into the space of metrics on ∂M, it has a series expansion in t. If $\overline{g} \in C^{m,\alpha}$ is asymptotically hyperbolic on a 4-manifold $\overline{M}$, for $m \geq 3$, then, as a function on the image of ∂M under the geodesic flow,

$$g_t = g_{(0)} + t^2 g_{(2)} + t^3 g_{(3)} + ... + t^m g_{(m)} + O(t^{m+\alpha})$$

where

$$g_{(j)} = \frac{1}{j!}(\mathcal{L}^{(j)}_{\overline{\nabla}t}\overline{g}) \mid_{\partial M},$$

the j^{th} Lie derivative of $\overline{g}$, w.r.t. $\overline{\nabla}t$, evaluated on ∂M.

For n-manifolds generally a similar expansion will exist, but high-order terms (relative to n) may not be powers of t, and their coefficients may not depend just on $\overline{g}$ near ∂M. Therefore, such expansions will generally be taken to finite order with a controlled difference term, as is shown in the 4-dimensional case. Details are in [1].

Correspondingly, if $\tau = -\log t$ then the integral curves of $\nabla\tau$ are geodesics, and the metric g has the form $g = d\tau^2 + g_\tau$. It is possible, through the function τ, to define renormalized quantities of volume, Einstein-Hilbert action and other integrated curvatures. This addresses, among other things, the difficulty mentioned in the section *Einstein Metrics.*

5. An Example

Both examples given so far have had the same geometry — that of flat Euclidean 3-space. The flat geometry corresponds to the solution of the Einstein equation in a vacuum, without a cosmological constant, therefore it is not an Einstein metric in the sense we are using the term. It would be worthwhile to construct a more realistic example. In fact, the example we construct will be relevant to the statement of one of the theorems in the section *Existence and Uniqueness.*

As before, we start with the unit ball. This construction works in any number of dimensions, but for concreteness we will use exactly three. Again, we parameterize by r and Ω. Whereas previously, if the metric was of the form

$$f^2(r)dr^2 + h^2(r)d\Omega^2$$

then we required that $g = \int r$, so that the angular coordinate scaled with the metric radius, this relation is not satisfied for

$$g = \frac{dr^2 + r^2 d\Omega^2}{(1-r^2)^2}.$$

This indicates curvature. One can check that, in fact, $Ric_g \propto -g$.

The metric can be expressed in Cartesian coordinates as well:

$$g = \frac{dx^2 + dy^2 + dz^2}{(1 - |(x,y,z)|^2)^2}$$

This is known as the Poincaré metric. (Sometimes the term *Poincaré metric* is used more generally.) It is the prototypical Einstein metric on the ball. The Poincaré metric is not merely asymptotically hyperbolic; it is hyperbolic everywhere.

A straightforward compactification suggests itself. Let $\rho = 1 - r^2$. Then the compactification is

$$\rho^2 g = dr^2 + r^2 d\Omega^2$$

— the Euclidean metric on the ball. The conformal structure at infinity is

$$[\gamma] = \left[d\Omega^2\right].$$

The construction the geodesic compactification of g has difficulties, but we proceed heedlessly: we note the spherical symmetry of g and conclude that shortest paths to infinity under the compactification will surely be radial. If

$$\overline{g} = \overline{f}^2(r)dr^2 + \overline{h}^2(r)d\Omega^2$$

then the defining function satisfies

$$t = \int_r^1 \overline{f}(q)dq.$$

But, by definition, $\overline{f} = t\frac{1}{1-r^2}$, so we have

$$\frac{dt}{dr} = -\frac{t}{1-r^2}.$$

We conclude that

$$t = \frac{1-r}{\sqrt{1-r^2}}$$

and

$$\overline{g} = \frac{(1-r)^2(dr^2 + r^2 d\Omega^2)}{(1-r^2)^3}.$$

The difficulties become clear: dt and $\overline{g}$ are apparently undefined near ∂M. We would be unjustified if we were to say infinite, since the tangent spaces at the singularity, on which they would act, are similarly unclear. Moreover, the line integral from a point to the boundary under $\overline{g}$ is *finite*, and corresponds to t. Then $\overline{g}$ definitely describes a compact manifold. The problem is that the geodesic compactification does not respect the manifold structure of $\overline{M}$ under our chart. An alternative chart, with its correspondingly different representation of g, will give the same defining function, but will have a metric smooth on $\overline{M}$.

However, it is not worth our while to pass into a more difficult chart for this sake, since we already have the geodesic defining function t. We notice that

$$dt^2 = \frac{(1-r)^2}{(1-r^2)^2}dr^2$$

and therefore $\overline{g}$ splits as expected. In fact, it is possible to take Lie derivatives of $\overline{g}$ in the limit approaching ∂M,

$$\frac{1}{j!}(\mathcal{L}^{(j)}_{\nabla t}\overline{g})\ |_{r\to 1},$$

which will be finite, and will give the series expansion of g_t in t.

6. Existence and Uniqueness

Let $E^{m,\alpha}_{AH}$ be the space of AH Einstein metrics on M for which there exists a compactification $\overline{g}$ in $C^{m,\alpha}$. Let $\mathcal{E}^{m,\alpha}_{AH}$ be the space of equivalence classes in $E^{m,\alpha}_{AH}$ by orientation-preserving diffeomorphisms which extend to the identity on ∂M under a compactification. $E^{m,\alpha}_{AH}$ and $\mathcal{E}^{m,\alpha}_{AH}$ can be given the $C^{m,\alpha'}$ topology if $\alpha' < \alpha$. The restriction of the equivalence relation on $E^{m,\alpha}_{AH}$ to only those diffeomorphisms that fix ∂M is made so that the following simplification can be made to the boundary data: let $\mathcal{C}^{m,\alpha}$ be the space of $C^{m,\alpha}$ *pointwise* conformal classes on ∂M, by which we mean simply that $\gamma \sim \gamma'$ if $\gamma = c\gamma'$ for some real-valued function c. $\mathcal{C}^{m,\alpha}$ derives a topology similarly.

Finally, let $\Pi^{m,\alpha} : \mathcal{E}^{m,\alpha}_{AH} \to \mathcal{C}^{m,\alpha}$ be the natural map from classes of metrics on M to their conformal infinities. It is now sensible to ask questions about the surjectivity and injectivity of $\Pi^{m,\alpha}$, corresponding to the existence and uniqueness respectively of a $[g]$ for each $[\gamma]$.

It happens that the case of dimension four is easiest treat, or at least it has been treated most extensively so far. Fortunately, this is the case most applicable to classical general relativistic physics, and it serves as a first step in extending the results eventually to higher dimensions. One supposes that if existence and uniqueness hold in four dimensions, then they will also hold in higher dimensions, up to some topological issues.

Unfortunately, uniqueness does not hold generally. A partial uniqueness result, due to Michael Anderson [And2], relates a finite number of the coefficients in the Taylor expansion of the geodesic compactification to the metric on the interior [3]:

Theorem 6.1. *If M is 4 dimensional then the data $(\gamma, g_{(3)})$ on ∂M uniquely determines $g \in E_A H$ up to local isometry.*

$g_{(3)}$ is the third-order coefficient in the series expansion of $\overline{g}$, but is here taken to be data. The qualification "local" is in fact a strength of the theorem. It encompasses the possibility that two different manifolds M^1 and M^2 have the same boundary, and same boundary data, in which case it asserts that they have a diffeomorphic universal cover, and that locally there are always isometries taking g^1 to g^2. In the case that their fundamental groups are equal, it reduces to the statement that they are isomorphic.

This theorem does not directly speak to the main uniqueness problem. The data $(\gamma, g_{(3)})$ is different and apparently stronger than a mere conformal class. There are examples known of conformal classes for which the interior metric is not unique up to isometry [3]. It is being studied how common these are. It is known that in any compact region of $\mathcal{E}$ there will only be finitely many compatible classes for a particular conformal infinity.

Let $\mathcal{C}^{m,\alpha}_{+}$ be the space of non-negative conformal classes in $\mathcal{C}^{m,\alpha}$. By this is meant that $[\gamma] \in \mathcal{C}^{m,\alpha}_{+}$ has a representative which is not flat and which has non-negative scalar curvature. Let $\mathcal{E}^{m,\alpha}_{AH+} = \Pi^{-1}(\mathcal{C}^{m,\alpha}_{+})$. Anderson [2] has the following result:

Theorem 6.2. *If M is a 4-manifold for which the inclusion $\iota : \partial M \to \overline{M}$ induces a surjection*

$$H_2(\partial M, \mathbb{R}) \to H_2(\overline{M}, \mathbb{R}) \to 0$$

then for any $m \geq 4$, α, the boundary map $\Pi : \mathcal{E}^{m,\alpha}_{AH+} \to \mathcal{C}^{m,\alpha}_{+}$ is proper with respect to the $C^{m,\alpha}$ topology.

(Under a proper map, the inverse images of compact sets are compact.) $H_2(\cdot, \mathbb{R})$ is the real second homology. The homological hypothesis on M is a technical requirement that avoids certain known pathological cases. The properness of the map, combined with computations of its *degree* in particular cases, can establish its surjectivity. Anderson carries this through in the nest theorem [2].

Theorem 6.3. *Let M be unit 4-ball; then $\partial M = S^3$. If $\mathcal{C}^0$ is the connected component of the non-negative $C^{m,\alpha}$ ($m \geq 2$) conformal classes containing $[d\Omega]$ on S^3, then any $[\gamma] \in \mathcal{C}^0$ is the conformal infinity of an AH Einstein in $\mathcal{E}^{m,\alpha}_{AH}$ metric on M.*

This theorem fixes the topology to that of the 4-ball, which is a significant weakness. The theorem is somewhat stronger than is stated, but in its generality involves other mathematics not described here. In fact, for such a metric g, $[g]$ will be in the connected component of $\Pi^{-1}(\mathcal{C}^0)$ which contains the Poincaré metric on the 4-ball. The argument for this theorem relies on the previous theorem.

For a suitably restricted class of conformal classes, then, the existence of a compatible Einstein metric is established, at least for the trivial topology in dimension four. There is currently work being done to refine the existence results, and to study further the degree of uniqueness.

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