Assignment 1

Let Ω be a bounded open set of $\mathbb{R}^d$. Suppose that for all $t \in [0,T]$, $\mathbf{X} : x \mapsto \mathbf{X}(t,x)$ is a smooth C^∞ mapping from $\Omega \subset \mathbb{R}^d$ to $\mathbb{R}^d$.

For all $\varphi \in C_c^\infty(\mathbb{R}^d)$ we define

$$< \rho(t), \varphi >= \int_\Omega \varphi(\mathbf{X}(t,x)) \; dx.$$

1- Show that for all $t \in [0,T]$, ρ is a distribution over $\mathbb{R}^d$ (i.e. $\rho \in \mathcal{D}'(\mathbb{R}^d)$), and show that

$$|< \rho(t), \varphi >| \leq |\Omega| \sup_{x \in \mathbb{R}^d} |\varphi(x)|.$$

2- Assuming that for all $t \in [0,T]$, $\mathbf{X}$ is a diffeomorphism from Ω to Ω' with Ω' also a bounded open set of $\mathbb{R}^d$, show that one can write

$$\forall \varphi \in C_c^\infty(\Omega'), \; < \rho, \varphi >= \int_{\Omega'} \bar{\rho}(y) \varphi(y) \; dy$$

for some function $\bar{\rho} : \Omega' \mapsto \mathbb{R}$, and express $\bar{\rho}$ in terms of $\mathbf{X}$ and its derivatives.

We define $\partial_t \rho$ by

$$\forall \varphi \in C_c^\infty(\mathbb{R}^d), \; < \partial_t \rho(t), \varphi >= \int_\Omega \nabla \varphi(\mathbf{X}(t,x)) \cdot \partial_t \mathbf{X}(t,x) \; dx.$$

3- Show that $< \partial_t \rho, \varphi >= \frac{d}{dt} < \rho, \varphi >$.

4- Show that $\partial_t \rho \in \mathcal{D}'(\mathbb{R}^d)$ and that

$$|< \partial_t \rho(t), \varphi >| \leq |\Omega| \sup_{y \in \mathbb{R}^d} |\nabla \varphi(y)| \sup_{x \in \Omega} |\partial_t \mathbf{X}(t,x)|.$$

5- Assume again that for all $t \in [0,T]$, $\mathbf{X}$ is a diffeomorphism from Ω to Ω' with Ω' also a bounded open set of $\mathbb{R}^d$.

a) Show that one can write

$$\partial_t \mathbf{X}(t,x) = \mathbf{v}(t, \mathbf{X}(t,x))$$

for some velocity field $\mathbf{v}(t) : \Omega' \mapsto \mathbb{R}^d$.

b) Show that for all $\varphi \in C_c^\infty(]0,T[\times\mathbb{R}^d)$, we have

$$\int_{[0,T]\times\mathbb{R}^d} \partial_t\varphi(t,x)\bar{\rho}(t,x) + \nabla_x\varphi(t,x)\cdot v(t,x)\bar{\rho}(t,x)dx = 0$$

with $\bar{\rho}$ as in question 2 (Do not use the explicit expression of $\bar{\rho}$ in terms of $\mathbf{X}$!!).

7- For all $t \in [0,T]$, define $J(t)$ by

$$\forall\phi \in \left(C_c^\infty(\mathbb{R}^d)\right)^d,\ <\phi, J(t)>= \int_\Omega \phi(\mathbf{X}(t,x))\cdot\partial_t\mathbf{X}(t,x).$$

Show that $J \in \left(\mathcal{D}'(\mathbb{R}^d)\right)^d$ and that

$$|<\phi,J>| \le |\Omega| \sup_{x\in\Omega}|\partial_t\mathbf{X}(t,x)|\ |<\rho(t),|\phi|>|.$$

8- Show that for all $\varphi \in C_c^\infty(]0,T[\times\mathbb{R}^d)$, we have

$$\int_{[0,T]} <\partial_t\varphi(t),\rho(t)> + <\nabla\varphi(t), J(t)>\ dt = 0.$$

9- Let $\rho \in C^1(\mathbb{R}^+\times\mathbb{R}^d)$, $\rho \ge 0$. Assume that there exists $C, C' > 0$ such that for all $t \in \mathbb{R}^+$, $\int_{\mathbb{R}^d}\rho(t,x)\ dx \le C$ and that for all (t,x), $\rho(t,x) \le C'$. Show that, for all $f \in C^1(\mathbb{R})$ with $f(0) = 0$, there exists C'' such that

$$\int_{\mathbb{R}^d} f(\rho(t,x))\ dx \le C'.$$

Let $\mathbf{v} \in \left(C^1(\mathbb{R}^+\times\mathbb{R}^d)\right)^d$, $\mathbf{v} = (v_1,\ldots,v_d)$. Suppose that

$$\sum_{i=1}^d \frac{\partial v_i}{\partial x_i} \equiv 0$$

(this is also denoted by $\nabla\cdot\mathbf{v} \equiv 0$). Suppose that $\rho, \mathbf{v}$ satisfies the equation

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0$$

(where $\nabla\cdot(\rho\mathbf{v}) = \sum_{i=1}^d \frac{\partial(\rho v_i)}{\partial x_i}$). Show that for all f as above we have

$$\frac{d}{dt}\int f(\rho(t,x))\ dx \equiv 0.$$