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## Partial Differential Equations: Application in Spatial Economics

### ABSTRACT

Partial differential equations have many applications in physics, geometry, and economics, just to name a few. An application in spatial economics is studied, focusing on the research of Martin Beckmann and Tõnu Puu. The flows of commodities in the space economy and the relevance of the transportation cost are defined. The problem of minimizing the transportation cost, known as the continuous transportation model and Beckmann's flow model is presented. The constraints of the problem are derived. The solution is produced using Euler-Lagrange equations. The application of the solution is examined and the usefulness of Beckmann's equations is briefly discussed. Economic and mathematical interpretations are used.

Spatial economics studies flows of commodities, people, money or information within a certain region. The focus will be on two dimensional flows of commodities between regions. The continuous analysis will be used as opposed to the discrete analysis since using spatial coordinates is more convenient. Spatial coordinates are given

by  $\underline{x} = (x, y)$ . Types of commodities, for example, include agricultural products, coal, wood, and so forth. For a given location there is a supply and a demand for commodities. It is assumed that prices vary between locations and commodity shipments occur from low-price to high-price areas. The equilibrium of spatial market within a closed region suffices to have a condition where aggregate supply equals aggregate demand. A perfect competition efficient allocation of resources will be considered.

The excess demand is the demand minus the supply. The excess demand density is a function of location and is denoted by  $q(x, y)$ . The movement of commodity proceeds from points of excess supply to points of excess demand. The mathematical interpretation for an equilibrium condition of the spatial market is

$$\iint_A q(x, y) dx dy = 0.$$

It is interesting to note that net exports which is exports (X) minus imports (M) is equal to  $-\iint_A q(x, y) dx dy$ . When transporting commodities between regions, a cost is required. Local transportation cost,  $k = k(\underline{x}) = k(x, y)$ , depends on location. The minimum transportation cost, denoted as  $\lambda(x)$  from  $\underline{x}^0$  to  $\underline{x}$  where

$$\lambda(x) = \min_{\underline{x}(s)} \int k(x(s), y(s)) ds.$$

The local flow of each commodity is represented as a vector as a function of location and is denoted by  $\phi(x, y)$ . The local flow of a commodity has a direction that moves through trade and a volume which is the quantity of objects being shipped. The commodity movement in interlocal trade is described by a continuous flow field. The flow field vanishes when there is no trade. Therefore, the supply equals the demand. The flow vector of commodities  $\phi$  is equal to  $\phi_1(x, y), \phi_2(x, y)$ . The volume of flow is equal to

$$|\phi| = \sqrt{\phi_1^2 + \phi_2^2}.$$

The direction of flow (the unit vector field) is

$$\phi/|\phi| = (\cos \theta, \sin \theta).$$

The relationship between the flow fields and excess supply (negative of excess demand) is

$$-q(x, y) = \text{div } \phi = \partial \phi_1 / \partial x + \partial \phi_2 / \partial y.$$

This is known as the equilibrium of quantity or the interregional trade equilibrium. When there is no flow across the boundary, or no exports/imports occurring, the flow vector outside and across the boundary is zero. So

$$\phi_n = 0 \text{ in } \partial A$$

where  $n$  is the direction normal to the boundary, pointed outward;  $\partial A$  is the boundary of an area  $A$ .

The continuous transportation model can now be presented. The problem is

$$\min_{\phi} \iint_A k |\phi| \, dx \, dy$$

where the transportation cost is  $\iint_A k |\phi| \, dx \, dy$ , measured in money terms or commodity units. This is subject to  $\text{div } \phi + q = 0$ . If on the boundary, only  $\phi_n = 0$  in  $\partial A$  is required. In order to solve this problem, Euler-Lagrange equations are needed.

Recall the derivation of Euler-Lagrange equations. There are two independent variables  $x$  and  $y$ . An optimizing function of these coordinates,  $u(x, y)$  is needed along with its partial derivatives. The result is the minimization of  $I = \iint_R F(x, y, u, p, q) \, dx \, dy$  where  $p = \partial u / \partial x$  and  $q = \partial u / \partial y$ . The boundary condition is  $u = u_0(x, y)$  on the curve  $C$ , where  $u_0$  is a given function. Assume a test function,  $\phi(x, y)$ , is a solution to the minimization problem. Then  $u(x, y) + \varepsilon \cdot \phi(x, y)$  must satisfy the boundary condition.

This requires that  $\varphi(x, y) = 0$  on  $C$ . Substitute  $u(x, y) + \varepsilon \cdot \varphi(x, y)$  into  $I = \iint_R F(x, y, u, p, q) dx dy$  and differentiate with respect to  $\varepsilon$ , and set the derivative to 0. So,

$$dI/d\varepsilon = \iint_R (\varphi F_u + \varphi_x F_p + \varphi_y F_q) dx dy = 0.$$

Consider the divergence and use the divergence theorem to get

$$\text{div}(\varphi F_p + \varphi F_q) = \varphi \partial/\partial x F_p + \varphi \partial/\partial y F_q + \varphi_x F_p + \varphi_y F_q.$$

Next, apply Gauss's Theorem to get  $\iint_R \text{div}(\varphi F_p + \varphi F_q) dx dy = 0$  since  $\varphi \equiv 0$  on the boundary curve  $C$ . So,  $\iint_R (\varphi_x F_p + \varphi_y F_q) dx dy = - \iint_R (\varphi \partial/\partial x F_p + \varphi \partial/\partial y F_q) dx dy$  is the result. Lastly, substitute this into  $dI/d\varepsilon = \iint_R (\varphi F_u + \varphi_x F_p + \varphi_y F_q) dx dy = 0$  to derive  $\iint_R (F_u - \partial/\partial x F_p - \partial/\partial y F_q) \varphi dx dy = 0$ . Since  $\varphi$  is an arbitrary function, by the fundamental lemma of variational calculus,

$$F_u - \partial/\partial x F_p - \partial/\partial y F_q = 0$$

holds everywhere and this is known as the Euler-Lagrange equation. In the two-dimensional case, there are two test functions  $u(x, y)$  and  $v(x, y)$  with two boundary conditions  $u = u_0(x, y)$  and  $v = v_0(x, y)$  and the partial derivatives of  $v$  are  $r$  and  $s$ . The result is two different Euler-Lagrange equations:

$$F_u - \partial/\partial x F_p - \partial/\partial y F_q = 0$$

and

$$F_v - \partial/\partial x F_r - \partial/\partial y F_s = 0.$$

The transportation cost,  $\iint_A k |\varphi| dx dy$  can now be minimized subject to the constraint,  $\text{div} \varphi + q = 0$ . Formulate the Lagrangean:  $k |\varphi| + \lambda (\text{div} \varphi + q)$ . Let the two functions be  $u = \varphi_1(x, y)$  and  $v = \varphi_2(x, y)$ . Notice that  $\text{div} \varphi = u_x + v_y$ . The Lagrangean becomes

$$k \sqrt{(u^2 + v^2)} + \lambda (p + s + z).$$

Now, apply the Euler-Lagrange equations derived above to get

$$k u / \sqrt{(u^2 + v^2)} - \partial \lambda / \partial x = 0$$

and

$$k v / \sqrt{(u^2 + v^2)} - \partial \lambda / \partial y = 0.$$

The partial derivatives of  $\lambda$  form the gradient of  $\lambda$ . The two equations together, noting that  $u = \phi_1$  and  $v = \phi_2$  results in one single vector equation:

$$k \phi / |\phi| = \text{grad } \lambda.$$

This is known as the condition for optimal spatial pricing and transportation. The price equilibrium in a spatially extended market is given by

$$|\text{grad } \lambda| \leq k$$

$$k \phi / |\phi| = \text{grad } \lambda \text{ wherever } \phi \neq 0$$

where  $\lambda$  is the price of the commodity. This means the gain from trade cannot exceed the cost of transportation.  $k \phi / |\phi| = \text{grad } \lambda$  says that the flow field direction is the same direction as the gradient of the price of the commodity traded. Also, the price increases at the same rate as the cost of transportation. So this means that commodities are shipped in the direction where the price indicates that the commodity is scarce.

The objective function of this problem is convex, the constraints are linear, there exists a feasible solution to  $\iint_A q(x, y) dx dy = 0$ , and the minimand is bounded for the region  $A$  in two-space.  $k \phi / |\phi| = \text{grad } \lambda$  with constraints  $\text{div } \phi + q = 0$  and  $\phi_n = 0$  in  $\partial A$  uniquely determines the directions of a flow field  $\phi$  that is a solution to the problem. This model of interregional trade was invented by Martin Beckmann in 1952. It is known as Beckmann's Flow Model. He showed that  $k \phi / |\phi| = \text{grad } \lambda$  was the Euler-Lagrange equation that minimized  $\iint_A k |\phi| dx dy$  and was subject to  $\text{div } \phi + q = 0$ .

Beckmann's equation,  $k \phi/|\phi| = \text{grad } \lambda$ , can be used to derive the flow direction field. This is obtained by taking the squares of both sides of  $k \phi/|\phi| = \text{grad } \lambda$ . Note that  $(\phi/|\phi|)^2 \equiv 1$ . The result is the partial differential equations from prices,

$$\lambda_x^2 + \lambda_y^2 = k^2$$

or

$$\partial^2 \lambda / \partial x^2 + \partial^2 \lambda / \partial y^2 = k^2.$$

This equation is analogous to the Hamilton-Jacobi equation. The flow lines can be derived by solving  $\lambda_x^2 + \lambda_y^2 = k^2$  and using the definition of the direction field,  $k \phi/|\phi| = \text{grad } \lambda$ . Consider a parameterized trajectory  $(x(t), y(t))$ . The partial differential equations can be transformed into ordinary differential equations,

$$dx / dt = \lambda_x (x,y)$$

and

$$dy / dt = \lambda_y (x,y).$$

The equation,  $\lambda_x^2 + \lambda_y^2 = k^2$  can also be used to generalize the transportation model. It is interpreted as the squared gradient of the price,  $\lambda$ , of a shipped commodity equaling the squared freight rate,  $k$ . If the same transportation system is used for all commodities then the freight rates of each commodity will only differ by a constant factor from differences in weight and bulkiness, for example. This slight change is denoted by  $k = \kappa h (x,y)$  for some constant  $\kappa$ . The equation then becomes

$$\lambda_x^2 + \lambda_y^2 = (\kappa h (x,y))^2.$$

For each commodity  $i$ , this is  $(\text{grad } \lambda_i)^2 = \kappa_i^2 h (x,y)^2$ . This equation is helpful in deducing the partial differential equation for a power of land rent obtainable from the  $i^{\text{th}}$  activity.

To summarize Beckmann's equations:

$$k \varphi / |\varphi| = \text{grad } \lambda$$

minimizes

$$\iint_A k |\varphi| \, dx \, dy$$

subject to the constraint

$$\text{div } \varphi + q(x, y) = 0.$$

Beckmann's flow model and the continuous transportation model are one of many models that involve the use of partial differential equations. These models also reveal the importance of Euler-Lagrange equations which is a necessary result in deriving the solution to the problem of minimizing the transportation cost.

## Works Cited

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