ON OPTIMAL PARTITION OF LARGE WORK FORCE INTO WORKING GROUPS

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1. Introduction

In this paper we deal primarily with the following problem. Suppose we have a large work force with individuals of varying skills. We would like to partion this large work force into groups of two. We are given that productivity of the pair is some function which depands on the skill levels of the two individuals in such a way that it depands more on skills of one than the other. This would mean that we would assign a manager in each group and the efficiency of the group would depand more on the skills of the manager. The problem is to partion the work force in a way that maximizes the total efficiency of the work force, or in other words to optimaly pair workers with managers.
It is the purpose of this paper to define rigorous formulation of the aforementioned problem and of its solution and to show that there exist only one such solution. We will also exibit some properties of the solution and we will end our paper by discussion of possible directions of further research into this problem.

2. Rigorous formulation of the problem

We begin with a rigorous definition of the problem. Here and on, X is a compact Hausdorff space.

Definition 2.1 (Admissible pairing)**.** Given a measure $\mu \in \mathcal{P}(X)$ we say that (ρ, τ, h) is an admissible pairing for μ whenever $\rho, \tau \in \mathcal{P}(X)$, $h : X \to X$ is a borel map such that $\tau = h_{\#}\rho$ and $\rho + \tau = 2\mu$. We denote the set of all admissible pairings by $\mathcal{G}_\mu$.

Definition 2.2 (Optimal pairing)**.** We say that $g \in \mathcal{G}_\mu$ is an optimal pairing for μ whenever the following supremum is attained for g:

$$\sup\left\{ \int_X P(x, h(x))\, d\rho \,|\, (\rho, \tau, h) \in \mathcal{G}_\mu \right\} \tag{2.1}$$

We will refer to value of the above supremum as optimal efficiency.

We are now in the position to state the main theorem of this paper

Theorem 2.3 (Existance and uniqueness of optimal pairing)**.** *Suppose that $P \in C^2([0,1]^2)$, $\frac{\partial^2 P}{\partial x \partial y} \geq 0$ and P is strictly convex in x. Further suppose that $\mu \in \mathcal{P}([0,1])$ is a measure absolutely continuous w.r.t. Lebegue measure on $[0,1]$. Then there exist unique (with h being unique up to measure zero) optimal pairing for μ.*

3. Formulation of the relaxed problem

Just as in the case of optimal transportation problem, we can formulate the weaker problem then the one above, and just as in the case of optimal transportation, this weaker problem will provide us with the answer to the stronger problem.
We begin by formulating the weak problem.

Definition 3.1 (Admissible competitor in the weak problem)**.** Given a probability measure $\mu \in \mathcal{P}(X)$ we say that $\gamma \in \mathcal{P}(X^2)$ is an admissible competitor in the weak problem for μ, whenever $\gamma[B \times X] + \gamma[X \times B] = 2\mu[B]$ for all Borel sets $B \subseteq X$. We denote the set of all admissible competitors in the weak problem for μ by Γ_μ.

Definition 3.2 (Weak optimal pairing)**.** We say that $\gamma_0 \in \Gamma_\mu$ is a weak optimal pairing, whenever the following supremum is attained at γ_0

$$J_\mu = \sup\left\{ \int_X P(x, y)\, d\gamma(x, y) \,|\, \gamma \in \Gamma_\mu \right\} \tag{3.1}$$

We refer to the value of the above supremum as weak optimal efficiency.

Notice that $\gamma \in \mathcal{P}(X^2)$ is a competitor in the weak problem for μ if and only if

$$\int_{X^2} \phi(x) + \phi(y)\, d\gamma(x, y) = 2 \int_X \phi(x)\, d\mu(x) \qquad \forall \phi \in C(X) \tag{3.2}$$

where $C(X)$ is the space of all continuous functions on X.

For all μ the set Γ_μ is obviously convex. It is also closed in the *weak** topology induced by $C(X^2)$. This is because when $\{\gamma_n\} \in \mathcal{P}(X^2)$ converges weakly then $1 = \int_X 1\, d\gamma_n(x,y) \to \int_X d\gamma$ and $\int_X [\phi(x) + \phi(y)]\, d\gamma_n(x,y) \to \int_X [\phi(x) + \phi(y)]\, d\gamma(x,y)$, hence $\gamma \in \Gamma_\mu$. Thus Γ_μ is compact by Alaoglu, since it is also bounded (it contains only probability measures), and we conclude that there exist measure $\gamma_0 \in \Gamma_\mu$ which is a weak optimal pairing.

4. Dual problem

Since our problem is so closely related to Optimal Transportation it is natural to suspect that there exist a problem dual to ours, just as in the case of optimal transportation. And indeed it is the case that dual problem exist and it provides us with a valuable tool to study uniqueness and properties of the solution to the optimal pairing problem.

Let $\Phi = \Big\{\phi \in C(X) \,|\, \phi(x) + \phi(y) \geq P(x,y)\Big\}$ and consider infimum

$$I_\mu = \inf \Big\{ 2 \int_X \phi\, d\mu \,|\, \phi \in \Phi \Big\} \tag{4.1}$$

We claim the following:

Theorem 4.1 (Existance of dual problem)**.** *Given I_μ and J_μ as above we have $I_\mu = J_\mu$.*

The proof of this claim will be based on the following theorem:

Theorem 4.2 (Fenchel-Rockafellar duality)**.** *Let E be a normed vector space, E^* its topological dual space, and Θ, Ξ two convex functions on E with values in $\mathbb{R} \cup \{+\infty\}$. Let Θ^*, Ξ^* be the Legendre-Fenchel transform of Θ, Ξ respectively. Assume that there exist $z_0 \in E$ such that*

$$\Theta(z_0) < +\infty, \quad \Xi(z_0) < +\infty, \text{ and } \Theta \text{ is continuous at } z_0$$

Then,

$$\inf_E \Big[\Theta + \Xi\Big] = \max_{z^* \in E^*} \Big[-\Theta^*(-z^*) - \Xi^*(z^*)\Big] \tag{4.2}$$

the proof of which you can find in [2].

We now proceed with the proof of theorem 4.1 similarly to the proof of duality in the case of optimal transportation as outlined in [2].

Proof. Let $E = C(X^2)$ then $E^* = M(X^2)$ where $M(X^2)$ is a space of signed measures of bounded total variation, by Riez representation theorem. We then let

$$\Theta : u \in C(X^2) \longmapsto \begin{cases} 0 & \text{if}\, u(x,y) \geq P(x,y) \\ +\infty & \text{otherwise} \end{cases}$$

$$\Xi : u \in C(X^2) \longmapsto \begin{cases} 2\int_X \phi(x)\, d\mu & \text{if}\, u(x,y) = \phi(x) + \phi(y) \\ +\infty & \text{otherwise} \end{cases}$$

Notice that Ξ is well defined since if $\phi(x)+\phi(y) = u(x,y) = \psi(x)+\psi(y)$ then $\phi(x)-\psi(x) = \psi(y)-\phi(y) = c$, which implies that $\phi(x) = c+\psi(x)$ and $\phi(x) = \psi(x) - c$, hence $c = 0$ and $\psi = \phi$.
We also see that conditions of theorem 4.2 apply, we simply need to take $z_0 \equiv const$ with large enough constant. Then proceeding similarly to the proof of duality for optimal transportation in [2] we conclude the proof of theorem 4.1. □

5. Uniqueness of the solutions

In this section we discuss how dual problem leads us to the proof of the uniqueness of solutions in the case $X = [0,1]$. From now on we assume that X is an interval $[0,1]$ with the usual topology, and that $P \in C^2([0,1]^2)$ and $\frac{\partial^2 P}{\partial x \partial y} \geq 0$.

First of all notice that if $\gamma_0 \in \mathcal{P}(X^2)$ is a weak optimal pairing, then the following supremum is also attained for γ_0:

$$\sup\Big\{ \int_X P(x,y)\, d\gamma \,|\, \gamma \in \mathcal{P}(X^2), \gamma[B \times X] = \gamma_0[B \times X], \\ \gamma[X \times B] = \gamma_0[X \times B] \quad \forall B\, \text{Borel} \subseteq X \Big\}$$

From the theory of optimal transportation [1] it is well known that for measure μ absolutely continuous w.r.t. Lebegue the above supremum is attained uniquely and that there exist unique a.e. w.r.t. ρ (and thus Lebegue as well) monotone function h which pushes forward ρ to τ, where $\rho[B] := \gamma_0[B \times X]$ and $\tau[B] := \gamma_0[X \times B]$ for all Borel subsets of X. In fact the support of γ_0 is equal to $\{(x, h(x)) \in X^2 \,|\, \text{almost all}\, x \in \mathit{supp}(\rho)\}$. In particular we deduce that $(\rho, \tau, h) \in \mathcal{G}_\mu$ and that it is an optimal pairing. Thus we infered the existance of optimal pairing.
Now we prove the following theorem

Theorem 5.1 (Existance of dual problem)**.** *The infimum I_μ is attained whenever μ is absolutely continuous w.r.t. to Lebegue.*

Proof. First notice that $\int_X \phi(x)\, d\mu = \int_X \phi(x) f(x)\, dx$ by Radon-Nikodym theorem, where $f \in L^1, f \geq 0$ since μ is a probability measure and absolutely continuous w.r.t. Lebegue. Also $B(0, \|P\|_\infty) \cap \Phi$ is a nonempty convex closed bounded subset of $L_\infty([0,1])$ and hence *weak** closed and hence compact by Alaoglu theorem. Therefore continuous linear functional $T : \phi \in L_\infty \mapsto \int_X \phi(x) f(x)\, dx$ attains its infimum on $B(0, \|P\|_\infty) \cap \Phi$. Lastly notice that infimum of T over Φ is the same as that over $B(0, \|P\|_\infty) \cap \Phi$, since we can cut off any function in Φ to make its norm equal to $\|P\|_\infty$ and it will still be in Φ. □

At this point we have the following result.

Theorem 5.2. *Let $\gamma_0 \in \Gamma_\mu$ be a weak optimal pairing. Suppose that $\phi(x) \in \Phi$ is a solution to the dual problem then $\phi(x) + \phi(y) = P(x, y)$, $\forall (x, y) \in supp(\gamma_0)$.*

Proof. Due to duality we have the following:

$$\int_X P(x,y)\, d\gamma_0 = \int_X 2\phi(x)\, d\mu = \int_X \phi(x)\, d\gamma_0(x,y) + \int_X \phi(y)\, d\gamma_0(x,y)$$

Therefore we must have

$$\int_X [\phi(x) + \phi(y) - P(x,y)]\, d\gamma_0 = 0$$

Since $\phi(x) + \phi(y) \geq P(x, y)$ we therefore have that $\phi(x) + \phi(y) - P(x, y) = 0\ \forall (x, y) \in supp(\gamma_0)$. □

Now, we are in the position to prove theorem 2.3

Proof. Let ϕ be any solution of the dual problem and let $H_\mu = \Big\{(x, y) \in X^2 \mid \phi(x) + \phi(y) = P(x, y)\Big\}$. Notice that for any weak optimal pairing $\gamma_0 \in \Gamma_\mu$ its support must be a monotone subset of H_μ. Now suppose that $h_\mu \subseteq H_\mu$ is a support of some weak pairing γ_0 then since $P(x, y) - \phi(x) \leq \phi(y), \forall (x, y) \in X^2$ and for $(x_0, y_0) \in supp(\gamma_0)$ we must have $P(x_0, y_0) - \phi(x_0) = \phi(y_0)$, therefore $\phi(y_0) = \sup_{x \in X}[P(x, y_0) - \phi(x)]$ and similarly $\phi(x_0) = \sup_{y \in X}[P(x_0, y) - \phi(y)]$. But this implies that ϕ is semi-concave and thus in fact is differentiable a.e.. And since function $f : (x, y) \mapsto P(x, y) - \phi(x) - \phi(y)$ obtains it maximum at (x_0, y_0) we obtain $\frac{d\phi}{dx} = \frac{\partial P}{\partial x}(x_0, y_0)$ a.e. by differentiating f w.r.t. x. However, because $P(x, y)$ is strictly convex, $\frac{\partial P}{\partial x}$ is strictly increasing as a function of x, hence we can solve for y_0 as a function of x_0 for almost all x_0. Therefore we conclude that if γ_0 is any weak optimal pairing and $(\rho, \tau, h) \in \mathcal{G}_\mu$ is a corresponding optimal pairing as described above, then h must be unique up to measure zero. But then it is easy to check that ρ and τ must be unique as well. $\square$

6. Conclusion

In conclusion we would like to address several possible directions in which further research could proceed. First of all, it would be interesting to study the qualitative properties of solutions and how they relate to the measure μ. Second, one could try proving the above results in the case when $X \subseteq \mathbb{R}^n$ and is not necessarily compact. One also could attempt to generalize the above result by considering partitions into groups of size other then 2, or maybe even into groups of variable size. Finally, it would be interesting to see whether the above formalism can be used to model other physical phenomina.

References

[1] R.J. McCann. Exact solutions to the transportation problem on the line. *Proc. Royal Soc. London* Ser. A 455, 1341-1380 (1999).

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