Exercise 2 :

① differentiate $/x_i$ $\quad \partial_t \omega + \nabla(v\omega) = 0$

$$\partial_i \, v.\nabla\omega = \partial_i \Big(\sum_k v_k \, \partial_k \omega\Big)$$
$$= \sum_k \partial_i v_k \, \partial_k \omega + v_k \, \partial_k \partial_i \omega$$
$$= (v.\nabla)(\partial_i \omega) + \sum_k \partial_i v_k \, \partial_k \omega$$

② $\quad \partial_t \rho + v.\nabla\rho = f.$

-> multiply by $p\rho^{p-1}$

-> $\quad \partial_t(\rho^p) + v.\nabla(\rho^p) = p\rho^{p-1} f.$

integrate over the domain

$$\partial_t \int \rho^p + \int v.\nabla(\rho^p) = p \int \rho^{p-1} f$$

* $\displaystyle \int v.\nabla(\rho^p) = \int \nabla.(v\rho^p) - \int \rho^p \, \nabla.v$

use $\nabla.v = 0$, $v.\partial\Omega = 0$ (boundary of the domain Ω)

or v periodic if $\Omega = \mathbb{T}^2$

find $\displaystyle \int v.\nabla(\rho^p) = 0.$

* Hölder's inequality : $\displaystyle \left|\int f \rho^{p-1}\right| \leq \|f\|_{L^p} \, \|\rho^{p-1}\|_{L^{p'}}$

$\dfrac{1}{p'} + \dfrac{1}{p} = 1 \quad \Rightarrow \quad p' = \dfrac{p}{p-1} \quad \Rightarrow \quad \|\rho^{p-1}\|_{L^{p'}} = \|\rho\|_{L^p}^{p-1}$

We find

$$\frac{d}{dt}\int|\rho|^p \leq p\|f\|_{L^p}\left(\int|\rho|^p\right)^{\frac{p-1}{p}}$$

$$\frac{d}{dt}\left(\int|\rho|^p\right)^{\frac{1}{p}} = \frac{1}{p}\frac{d}{dt}\int|\rho|^p\cdot\left(\int|\rho|^p\right)^{\frac{1-p}{p}}$$

we conclude

$$\frac{d}{dt}\|\rho\|_{L^p} \leq \|f\|_{L^p}.$$

3) From question 1) we have

$$\frac{d}{dt}\|\partial_i\omega\|_{L^p} \leq \left\|\sum_{i,j=1}^{2}\partial_i v_j\,\partial_j\omega\right\|_{L^p}$$

$$\leq \left(\sum_{i,j=1}^{2}\|\partial_i v_j\|_{L^\infty}\right)\left(\sum_{j=1}^{2}\|\partial_j\omega\|_{L^p}\right)$$

Check that $|\nabla\omega| = \left(\sum|\partial_j\omega|^2\right)^{1/2}$

satisfies

$$c_1\sum|\partial_j\omega| \leq |\nabla\omega| \leq c_2\sum|\partial_j\omega|$$

and conclude that $\|\nabla\omega\|_{L^p} \leq c_2\sum\|\partial_j\omega\|_{L^p}$

$\|\nabla\omega\|_{L^p} \geq c_1\sum\|\partial_j\omega\|_{L^p}$

$\Rightarrow$ We find $\displaystyle\frac{d}{dt}\|\nabla\omega\|_{L^p} \leq C\left(\sum_{i,j=1}^{2}\|\partial_i v_j\|_{L^\infty}\right)\|\nabla\omega\|_{L^p}$

Ex2, suite

4) b) implies that $\|D^2\psi\|_{L^p} \leqslant C_p \|\Delta\psi\|_{L^p}$

$$\|D^2(\nabla\psi)\|_{L^p} \leqslant C_p \|\Delta(\nabla\psi)\|_{L^p}$$

$$\left(\text{i.e. } \|D^2 \partial_i\psi\|_{L^p} \leqslant C_p \|\Delta \partial_i\psi\|_{L^p}\right). \ \forall)$$

thus from a), for $p>2$

$$\|D^2\psi\|_{C^\alpha} \leqslant C_p \left(\|D^2\psi\|_{L^p} + \|D^2\nabla\psi\|_{L^p}\right)$$

$$\leqslant C_p \left(\|\Delta\psi\|_{L^p} + \|\Delta\nabla\psi\|_{L^p}\right)$$

from *

$\nabla^\perp\psi = v$, thus

$$\sum \|\partial_j v_i\|_{L^\infty} \leqslant C\|D^2\psi\|_{L^\infty} \leqslant C\|D^2\psi\|_{C^\alpha}$$

and $\Delta\psi = \omega$ thus

$$\|D^2\psi\|_{C^\alpha} \leqslant C_p \left(\|\omega\|_{L^p} + \|\nabla\omega\|_{L^p}\right)$$

for $p>2$.

We conclude $\sum \|\partial_j v_i\|_{L^\infty} \leqslant C_p' \left(\|\omega\|_{L^p} + \|\nabla\omega\|_{L^p}\right)$

and since $\int \omega = 0$ on $\mathbb{T}^2$

we have also $\|\omega\|_{L^p} \leqslant C_p'' \|\nabla\omega\|_{L^p}$

$\to$ . . .

5) Using this result in 3) we get

$$\frac{d}{dt} \|\nabla w\|_{L^p} \leqslant C_p'' \|\nabla w\|_{L^p}^2 \quad \text{for some } C_p''$$

6) If $f_2(0) \leqslant f_1(0)$

and $\dot{f}_1 = c f_1^2$; $\dot{f}_2 \leqslant c f_2^2$

then for all t (as long as both are defined)

we have $f_2(t) \leqslant f_1(t)$

the solution of $\dot{f}_1 = c f_1^2$ is

$$f_1(t) = \frac{1}{T^* - ct} \quad \text{on } t \in [0, T^*[$$

with $T^* = (f_1(0))^{-1}$.

Thus from 5) we have

$$\|\nabla w\|_{L^p}(t) \leqslant \frac{1}{T^* - C_p'' t}$$

$$T^* = \frac{1}{\|\nabla w\|_{L^p}(0)}.$$