ALEXANDROFF

Existence and Uniqueness of a Convex surface with a Given Integral Curvature

$F = \partial C$, $C \subset \mathbb{R}^n$ a convex body

For a subset $E \subset F$, the Gauss map ω takes E to its "spherical image" $\omega(E)$ on the unit S^{n-1} sphere
Area $\phi(E)$ $\equiv$ integral curvature of E

Integral curvature defines a measure on F, (of total mass 4π if C is bounded and $n=3$). This measure is pushed forward to Haar measure on S^{n-1} by the Gauss map.

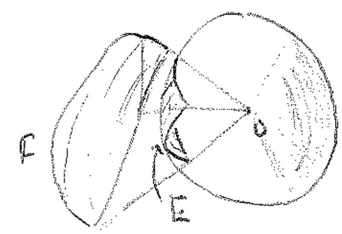
To compare the integral curvature on two different surfaces F, F', we position them so both enclose the origin O and then project the measure onto S^{n-1}. Moreover, Alexandroff wishes to start with a (correctly normalized) measure on S^{n-1} and ask whether a surface F with this integral curvature may be constructed.

Proofs are given for $n=3$ but quite general. Cases C bounded, unbounded are handled separately.

EXISTENCE **THEOREM 1:** Given a $\oplus^{ve}$ Borel measure k on S with
i) total mass 4π $(n=3)$
ii) for any convex subset $E \subset S^{n-1}$, $k(E) < 4\pi - \psi$
where $\psi =$ ~~Vol (U+E)~~ Haar (E^{o}) polar of E
there exists a closed convex surface F with k as its "reduced" (i.e. projected to S^{n-1}) integral curvature

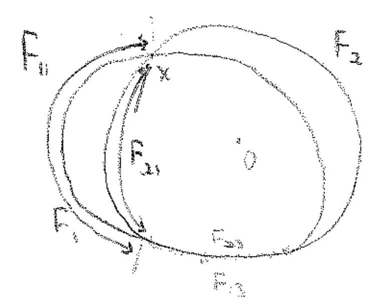
the condition ii) $k(E) < 4\pi - \text{Haar}(E^\circ)$ ← polar

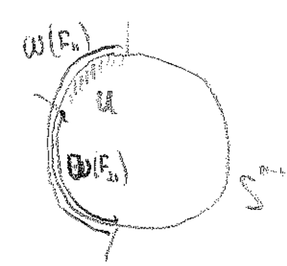
arises because F is supposed to enclose O. which can't happen otherwise



UNIQUENESS THEOREM 2: Let F_1 and F_2 be closed (n-1 dim'l) convex surfaces enclosing $O \in \mathbb{R}^n$, whose integral curvatures project to the same measure on S^{n-1}. Then F_1 is a dilation of F_2.

PROOF: Integral curvature is invariant under dilation. Choose a dilation of F_2 such that the surfaces intersect at some x. If the surfaces do not coincide then one of them, say F_1, has points F_{11} lying outside the other. Also F_2 has points lying inside F_1, denoted F_{21}. These point sets are open. The points where F_1 and F_2 coincide will be denoted $F_{13} = F_{23}$, $\ni x$ where the extra subscript will be used to indicate which of the two Gauss maps ω we are intending to apply.





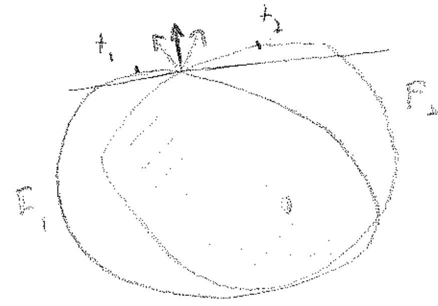
1) $\omega(F_{11}) \supset \omega(F_{21})$ showing the inclusion is strict in the sense of measures on S^{n-1}

2) ω takes closed sets to closed sets. Applied to $\sim F_{11}$ and $F_{21} \cup F_{23}$, this shows
$U = [S^{n-1} \sim \omega(\sim F_{11})] \cap [S^{n-1} \sim \omega(F_{21} \cup F_{23})]$ to be open

3) Assume x was chosen so both F_1 and F_2 have a

smaller, to verify

do not coincide*
The bisection is also
tangent to the intersection
$C_1 \cap C_2$ (where $F_i = \partial C_i$),
Hence the points t_1 and t_2
with the same tangent respectively
F_1 and F_2 lie $t_1 \in F_{11}$
$t_2 \notin F_{21} \cup F_{23}$
so $\omega(t_1) \in U$.

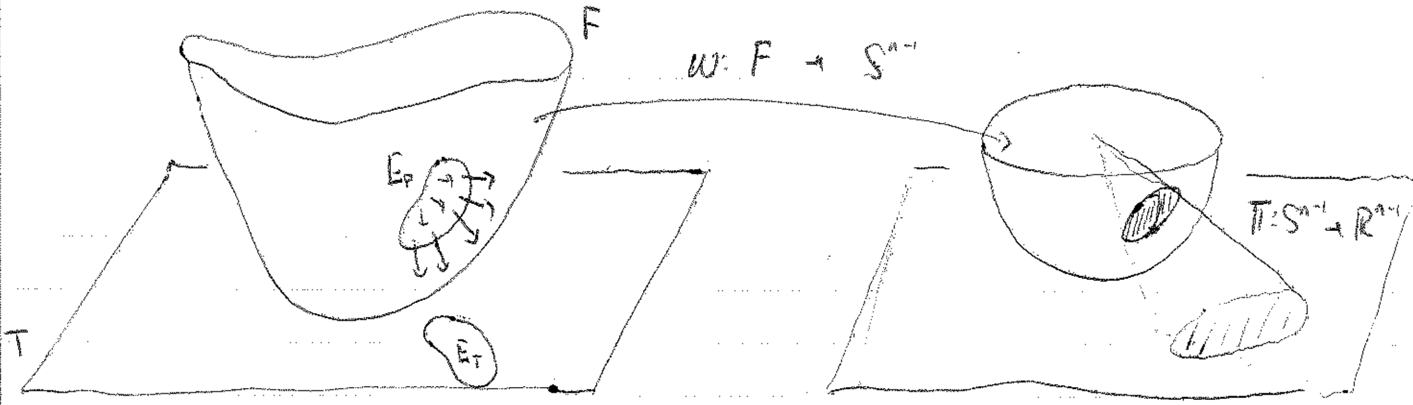


Infinite Complete Convex surface ICCS

By ICCS, I think Alexandroff means essentially the graph of a proper convex function of $\mathbb{R}^{n-1}$
which does not jump to ∞ ie is continuous to $\mathbb{R}$

T is some convenient $n-1$ hyperplane in $\mathbb{R}^n$ on which the function F might be defined ... say any supporting hyperplane.

(since Alexandroff is thinking geometrically, he wants to consider $F = \mathrm{bd}\, C$ for some unbounded convex C, but needs to explicitly exclude "cylinders" (C contains a line) which would correspond to improper convex functions)



note ~~[illegible]~~

$$\begin{tikzcd} F \arrow[r, "\omega"] & S^{n-1} \arrow[d, "\pi"] \\ T = \mathbb{R}^{d-1} \arrow[u, "F"] \arrow[r, "\nabla F"] & \mathbb{R}^{d-1} \end{tikzcd}$$

$\pi_\# \mathrm{Haar} = \dfrac{d^{n-1}x}{(|x|^2+1)^{n/2}}$

I think

In any case, the projection (orthogonal) onto T of the integral curvature measure of F is exactly the measure k on T for which

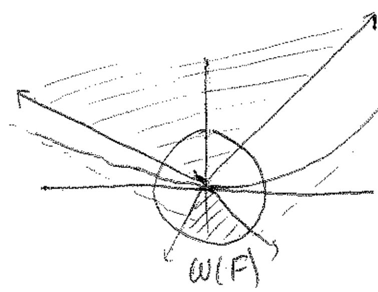
$$(\nabla F)_\# k = \pi_\# \mathrm{Haar}\Big|_{\mathrm{Im}\,\nabla F} = \frac{d^{n-1}x}{(|x|^2+1)^{n/2}}\Big|_{\mathrm{Im}\,\nabla F},$$

~~at least on $\mathrm{Im}\ \nabla F$~~

EXISTENCE

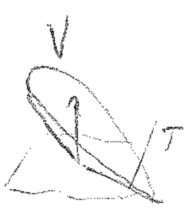
THEOREM 3: Any positive Borel measure k on $\mathbb{R}^{n-1}$ corresponds to the integral curvature measure for some convex F $\iff$ $k(T=\mathbb{R}^{n-1}) \leq 2\pi$

"limit cone of F": the convex cone whose spherical image coincides with that of F; ie the polar $\omega(F)^\circ$



+ note: convexity on the $n-1$ sphere is the same as convexity for the projection onto $\mathbb{R}^{n-1}$, because great circles (geodesics) are 'defined' as the intersection $S^{n-1} \cap P$ - 2 plane thru 0, which of course projects to $\{x \in \mathbb{R}^n ; x_n = -1\} \cap P$ ie a line.

EX & UNIQUENESS



THEOREM 4: Given a positive Borel measure k ~~...~~ on $\mathbb{R}^{n-1}$ and a convex cone V (containing a half-line perpendicular to $T=\mathbb{R}^{n-1}$) whose spherical image has volume $k(T) \leq 2\pi$. Then there exist a unique (up to translation) convex function F of T with integral curvature k and limit cone V

ie. the limit cone V selects a convex subset of $\mathbb{R}^{n-1}$ (V°) ~~...~~ with the correct $\frac{d^{n-1}x}{(|x|^2+1)^{n/2}}$ measure to be Im of F.

In the case where $k(T) = 2\pi$ this must be the whole plane ($V = +e_n$ a ray ...)

UNIQUENESS

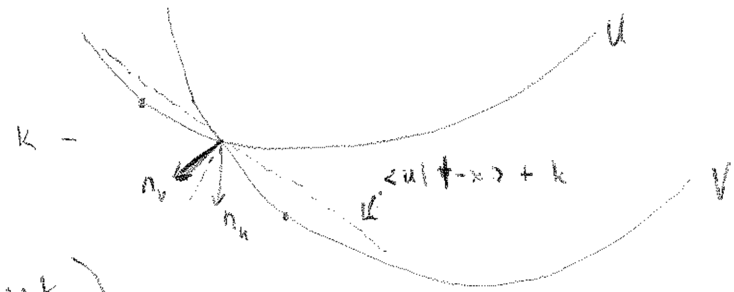
PROOF: Suppose there are two functions $u, v : T \to \mathbb{R} \cup \{\infty\}$ (if $\dim K = n$ we can rotate K to exclude ∞ as a value) with limit cone V and integral curvature K.

+ being locally Lipschitz

If $\nabla u = \nabla v$ everywhere they are both uniquely determined on $\operatorname{dom} u \cap \operatorname{dom} v \supset$ smooth K, then u and v can be reconstructed there by integration, and must agree up to a constant. Since neither can jump to ∞, $\operatorname{dom} u = \operatorname{dom} v$, hence $u = v$. The alternative is $\exists\, x$ s.t. $\nabla u(x) \neq \nabla v(x)$ and both exist. Shift v so that $u(x) = v(x)$.

WLOG suppose $T_1 = \{x : u < v\}$ is (open) non-empty. Use ω_i to denote the Gauss map composed with F_i. Clearly $\omega_1(T_1) \supset \omega_2(T_1)$, with the possible (& negligible) exception of extremal points of $\omega(V)$. If $T_3 = \{x : u = v\}$, then
$U = \sim [\omega_1(\sim T_1) \cup \omega_2(T_1 \cup T_3)]$
is relatively open in int $\omega(V)$.
Any $u \in U \Rightarrow u \notin \omega_2(T_1)$ and $u \in \omega_1(T_1)$
which $\Rightarrow \omega_1(T_1) \sim \omega_2(T_1) \supset U$ = positive measure
$\Rightarrow K(T_1) = \text{Area}(\omega_i(T_1))$ $\rightarrow\leftarrow$

Now $u = \frac{1}{2}(\nabla u(x) + \nabla v(x)) \in \operatorname{int} \Gamma(\omega(V))$ hence is in the subdifferential of both u and v. Note that the supporting hyperplanes to u and v at x intersect over the set $D = \{ t \mid \langle \nabla u(x) - \nabla v(x) \mid t - x \rangle = 0 \}$
Suppose $u \in \partial v(z), \partial u(y)$.
Monotonicity $\Rightarrow$ $\langle u - \nabla v(x) \mid z - x \rangle \geq 0$
$\langle \nabla u(x) - u \mid y - x \rangle \leq 0$
so z, y lie on opposite sides of D.



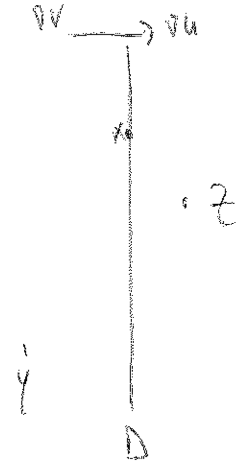
Finally,

$$v(z) < \langle u | z - x \rangle + k \quad \text{(using differentiability of } v\text{)}$$

since $z \in D$ $\quad \leq \langle \nabla u(x) | z - x \rangle + k$

$$\leq u(z)$$

so $z \notin T_1 \cup T_3$



$$u(y) < \langle u | y - x \rangle + k$$
$$\leq \langle \nabla v(x) | y - x \rangle + k$$
$$< v(y)$$

so $y \in T_1$

And, although it is possible that $\partial u(y)$ or $\partial v(z)$ are non-unique, any other point with the same gradient must be joined by a plane parallel to & below $\langle u | z - x \rangle + k$, hence satisfy the same strict inequality. Thus $u \notin \omega_1(\partial T_1)$ and $u \notin \omega_2(T_1 \cup T_3)$. → Clearly such a line cannot extend as far as D, because $\langle u | 0 - x \rangle + k$ is a lower bound for both u and v.

EXISTENCE in the POLYHEDRAL case

THEOREM 5: Given $K = \sum^{n} k_i \, \delta_{A_i}$, with $K(\mathbb{R}T) = \sum k_i \le 2\pi$ and a polyhedral angle V containing a ray perpendicular to T with Area(spherical image) $= K(T)$. Then there exists a convex polyhedral function with exactly n vertices, lying over $A_1 \dots A_n \in T$, and curvature k_i at A_i.

PROOF (SLICK!!):

CASE: $\sum k_i = 2\pi$, $\Rightarrow$ $V = +\hat{e}_n$

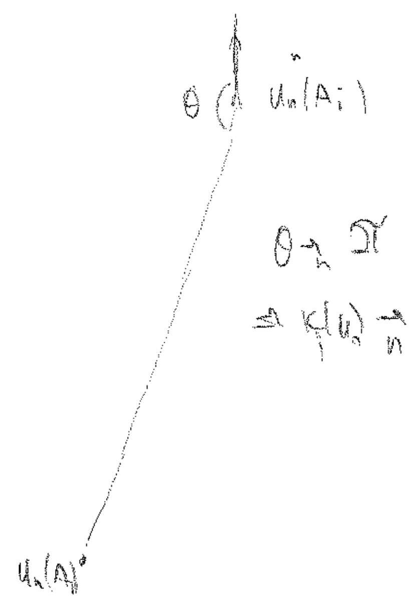
(I believe there may be subtleties in the other case which Alexandrov does not address: namely, one may require a consistency condition between V and the A_i for any polyhedral function with vertices exactly over the A_i to exist and also the absence of a boundary on the manifold is not clear to me).

Fix $u(A_1) = 0$. Any values for $u(A_i)$ $i = 2, \dots n$ generate a convex hull together with the ray V. Define $k_i(u(A_2) \dots u(A_n))$ to be the curvature at each vertex (which can only be one of the A_i). Conventionally $k_i = 0$ if $u(A_i)$ not on the surface of the convex body. k_i is a continuous function of the $u(A_i)$, so $k_i > 0$ $\forall i$ yields an $n-1$ dim'l manifold.

Also, the set of k_i s.t. $\sum k_i = 2\pi$, $k_i > 0$ is an $n-1$ dim'l manifold. The obvious map $u \to \{u(A_i)\}_{i=1}^{n}$ is a continuous map between these manifolds, and is one-one by our uniqueness result, therefore an open mapping by the invariance of domain theorem

Since the target manifold is connected, unless the image is onto, there is a point $\{k_i\}_i$ on its boundary. Choose a sequence of functions u_n with curvatures $\underset{n}{\to} \{k_i\}$.

The only way u_n can avoid having a convergent subsequence (a contradiction) is that either $K_i(u_n) \underset{n}{\to} 0$ (also contradicting $K_i(u_n) \to k_i > 0$.), or $u_n(A_i) \to \pm\infty$. Little circle forces $K_i(u_n) \to 0$ or $K_j(u_n) \to 0$ □



$\theta \underset{n}{\to} 2\pi$

$\Rightarrow K_j(u_n) \underset{n}{\to} 0$